COMBINATORIAL IDENTITIES FOR THE BI-PERIODIC FIBONACCI NUMBERS SQUARED

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ABSTRACT. In this paper, we provide a combinatorial interpretation for the bi-periodic Fibonacci numbers squared in terms of weighted linear tilings involving two types of tiles. This interpretation will allow us to establish combinatorial proofs for various identities concerning bi-periodic Fibonacci numbers squared.

1. INTRODUCTION

The Fibonacci sequence is a well-known example of a second-order recurrence sequence, defined by the recurrence formula $F_n = F_{n-1}+F_{n-2}$, with initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence and its generalizations have many interesting properties and applications in almost every field. The bi-periodic Fibonacci sequence was introduced by Edson and Yayenie in [3] as a generalization of the Fibonacci sequence. This sequence is defined for any integer $n \geq 2$ by the following recurrence relation

$$q_n = a^{\xi(n-1)}b^{\xi(n)}q_{n-1} + q_{n-2},$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero real numbers and $\xi(n) = n-2\lfloor n/2\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Note that for $a = b = 1$, we get the classical Fibonacci sequence.

The Fibonacci numbers can be interpreted as the number of ways to tile an n-board with squares and dominoes (see [2]). In [4], Edwards introduced a new type of tile, called a fence tile, which gives a new tiling interpretation of the Tribonacci numbers using squares and $(\frac{1}{2}, 1)$-fence tiles. A (w, g)-fence tile is composed of two pieces (called posts), a left fence post and a right fence post of size w, separated by a gap of size g. Edwards and Allen [6] show that the number of ways to tile an n-board using half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences is a Fibonacci number squared, F_{n+1}^2.

Our aim is to define the bi-periodic Fibonacci numbers squared and give a combinatorial interpretation using a weighted tilings approach, as well as provide several combinatorial proofs of some identities. Also, we discover new identities for the classic Fibonacci numbers squared.

Definition 1.1. *For any two nonzero real numbers a and b, and for any integer $n \geq 3$, the bi-periodic Fibonacci numbers squared, q_n^2, are recursively defined by*

$$q_n^2 = \begin{cases} \dfrac{a}{b}(ab+1)\,q_{n-1}^2 + (ab+1)\,q_{n-2}^2 - \dfrac{a}{b}q_{n-3}^2, & \text{if } n \text{ is even;} \\ \dfrac{b}{a}(ab+1)\,q_{n-1}^2 + (ab+1)\,q_{n-2}^2 - \dfrac{b}{a}q_{n-3}^2, & \text{if } n \text{ is odd;} \end{cases} \tag{1.1}$$

with the initial values $q_0^2 = 0$, $q_1^2 = 1$, and $q_2^2 = a^2$.

Note that for $a = b = 1$, we obtain the classic Fibonacci numbers squared, F_n^2.

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2. Weighted Tilings with Fences and Half-Squares

Consider a tiling board of length n with cells labeled from 1 to n, comprising weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences. A half-square is a tile with dimension $\frac{1}{2} \times 1$, whereas a $(\frac{1}{2}, \frac{1}{2})$-fence tile is composed of two fence posts of size $\frac{1}{2}$ separated by a gap of size $\frac{1}{2}$. In this tiling, if a half-square is placed in an odd cell i $(1 \leq i \leq n)$, it is assigned a weight of a. Conversely, if the half-square is placed in an even cell i, it is assigned a weight of b. Figure 1 illustrates the tilings and the product of their weights on a 3-board. As a result, we obtain $q_4^2 = a^4b^2 + 4a^3b + 4a^2$.

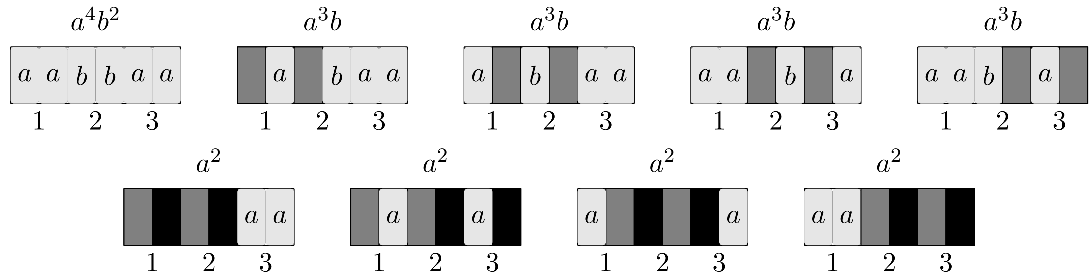

Figure 1. Different ways to tile 3-boards.

Any tiling can be expressed as a tiling with metatiles [4]. A metatile is a minimal arrangement of tiles that exactly covers an integral number (positive integer) of adjacent cells [4, 5]. In [6], the authors determined the types of metatiles with half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences. Similarly, we determine the types of metatiles with weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences.

The simplest metatile of length 1 consists of two adjacent free half-squares. It contributes b^2 to the weight when placed in an even cell (Figure 2 (1)) and a^2 to the weight when placed in an odd cell (Figure 3 (7)). The simplest metatile of length 2 is either the bifence with a weight of 1, which consists of two interlocking fences, or the filled fence with an additional weighted half-square added to one end, contributing ab to the weight (Figure 2 (2), Figure 3 (8)). The filled fence is obtained by filling the gap in the fence with a weighted half-square.

For metatiles of lengths $l \geq 3$, there are two types of mixed metatiles, namely, those that contain both half-squares and fences. To create metatiles of length $2(j + 1)$, we insert j bifences between a weighted half-square and a filled fence, which contributes ab to the weight (Figure 2 (4), Figure 3 (10)). Combining two filled fences gives a metatile of length 3, and inserting j bifences between them gives a metatile of length $2j + 3$. This metatile has a weight of b^2 when the end of each metatile is placed in an even position (Figure 2 (5)) and a^2 when placed in an odd position (Figure 3 (11)). Similarly, inserting j bifences between two half-squares produces a metatile of length $2j+1$, contributing b^2 to the weight when the end of each metatile is placed in an even position (Figure 2 (6)) and a^2 when placed in an odd position (Figure 3(12)).

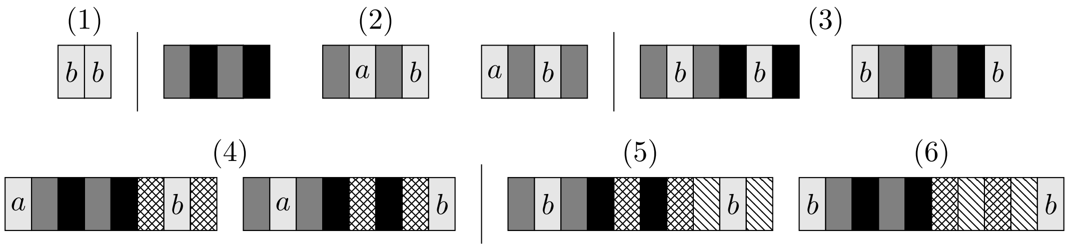

FIGURE 2. Types of metatile using $(\frac{1}{2}, \frac{1}{2})$-fences and weighted half-squares, where the end of each metatile is placed in an even position.

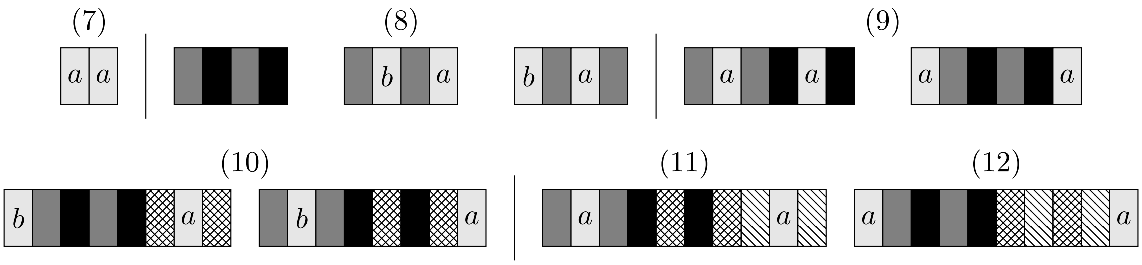

FIGURE 3. Types of metatile using $(\frac{1}{2}, \frac{1}{2})$-fences and weighted half-squares, where the end of each metatile is placed in an odd position.

For an integer $n \geq 0$, let S_n be the number of ways to tile an n-board using weighted half-squares and fences.

Lemma 2.1. *We have*

$$S_n = \begin{cases} \delta_{n,0} + b^2 S_{n-1} + (2ab+1)S_{n-2} + 2ab\sum_{j=2}^{n/2} S_{n-2j} + 2b^2 \sum_{j=1}^{n/2-1} S_{n-2j-1}, & \text{if } n \text{ is even;} \\ a^2 S_{n-1} + (2ab+1)S_{n-2} + 2ab\sum_{j=2}^{(n-1)/2} S_{n-2j} + 2a^2 \sum_{j=1}^{(n-1)/2} S_{n-2j-1}, & \text{if } n \text{ is odd;} \end{cases} \tag{2.1}$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise, and $S_n = 0$ for $n < 0$.

Proof. The result is obtained by a conditioning process on the last metatile. Consider a board of length n ($n \geq 1$). In the case where n is even, the number of ways to tile the board depends on the length of the last metatile:

- When the last metatile has a length of 1, there are $b^2 S_{n-1}$ possible ways to tile an $(n-1)$-board.
- When the last metatile has a length of 2, there are three types of metatiles: the bifence and the filled fence with adding a half-square to one end. In this case, there are $(2ab+1)S_{n-2}$ ways to tile an $(n-2)$-board.
- If the last metatile has a length of $l \geq 3$, there are two types of metatiles to consider. In this case, there are either $2b^2 S_{n-2j-1}$ ($j \geq 1$) or $2abS_{n-2j}$ ($j \geq 2$) ways to tile the $(n-l)$-board, where $l \leq n$.

Similarly, in the case where n is odd, there are $a^2 S_{n-1}$ ways to tile the $(n-1)$-board, $(2ab+1)S_{n-2}$ ways to tile the $(n-2)$-board, and for metatiles of length $l \geq 3$, the $(n-l)$-board can be tiled in either $2a^2 S_{n-2j-1}$ ($j \geq 1$) or $2abS_{n-2j}$ ($j \geq 2$) ways.

We let $S_0 = 1$ to count the empty board, and $S_{n<0} = 0$, because there is no way to tile an n-board with a metatile of length $l > n$. □

Theorem 2.2. *For any integer $n \geq 0$, q_{n+1}^2 counts the number of ways to tile an n-board using weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences.*

Proof. We show that $S_n = q_{n+1}^2$. For n even, by substituting n with $n-1$ in (2.1) and multiplying both sides by $\frac{b}{a}$, we get

$$\frac{b}{a} S_{n-1} = abS_{n-2} + \frac{b}{a} S_{n-3} + 2ab \sum_{j=2}^{n/2} S_{n-2j} + 2b^2 \sum_{j=1}^{n/2-1} S_{n-2j-1}. \tag{2.2}$$

Subtracting (2.2) from (2.1) yields

$$S_n = \delta_{n,0} + \frac{b}{a}(ab+1) S_{n-1} + (ab+1) S_{n-2} - \frac{b}{a} S_{n-3}.$$

For n odd, by substituting n with $n-1$ in (2.1) and multiplying both sides by $\frac{a}{b}$, we get

$$\frac{a}{b} S_{n-1} = \frac{a}{b}\delta_{n,1} + abS_{n-2} + \frac{a}{b} S_{n-3} + 2ab \sum_{j=2}^{(n-1)/2} S_{n-2j} + 2a^2 \sum_{j=1}^{(n-1)/2} S_{n-2j-1}. \tag{2.3}$$

Subtracting (2.3) from (2.1) yields

$$S_n = -\frac{a}{b}\delta_{n,1} + \frac{a}{b}(ab+1) S_{n-1} + (ab+1) S_{n-2} - \frac{a}{b} S_{n-3}.$$

Therefore, $S_0 = 1$, $S_1 = a^2$, $S_2 = (ab+1)^2$, and for $n \geq 3$, we have

$$S_n = \begin{cases} \frac{b}{a}(ab+1) S_{n-1} + (ab+1) S_{n-2} - \frac{b}{a} S_{n-3}, & \textit{if } n \textit{ is even;} \\ \frac{a}{b}(ab+1) S_{n-1} + (ab+1) S_{n-2} - \frac{a}{b} S_{n-3}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Using (1.1), we deduce that $S_n = q_{n+1}^2$. □

3. Combinatorial Identities

In the following section, we give some combinatorial identities of the bi-periodic Fibonacci numbers squared.

Identity 3.1. *For any integer $n \geq 0$, we have*

$$q_{n+1}^2 = \left(\frac{a}{b}\right)^{\xi(n)} \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j}{i} (ab)^{n-i-j}. \tag{3.1}$$

Proof. From Theorem 2.2, q_{n+1}^2 counts the number of ways to tile an n-board with weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences. On the other hand, we will use the following terminology: a free half square $(\frac{1}{2} \times 1)$, a filled fence $(\frac{3}{2} \times 1)$, and a bifence (2×1). In this context, let j denote the number of bifences in the tiling of an n-board, and let $i-j$ be the count of filled fences for some $0 \leq j \leq i$. Consequently, there will be $2n-3i-j$ free weighted half-squares. The total weights will be determined by $(ab)^{n-i-j}$ when n is even, and by $\frac{a}{b}(ab)^{n-i-j}$ when n is odd. These tilings, consisting of $2n-2i-j$ tiles, have a cardinality of

$$\left(\frac{a}{b}\right)^{\xi(n)} \binom{2n-2i-j}{j, i-j, 2n-3i-j} (ab)^{n-i-j} = \left(\frac{a}{b}\right)^{\xi(n)} \binom{i}{j} \binom{2n-2i-j}{i} (ab)^{n-i-j}.$$

Summing over i and j gives the identity. □

Note that for $a = b = 1$, we get the following identity (see [9])

$$F_{n+1}^2 = \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j}{i}.$$

Identity 3.2. *For any integer $n \geq 0$, we have*

$$q_{n+1}^2 = \left(\frac{a}{b}\right)^{\xi(n)} \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-3i}{j} (ab)^{n-i-j} (ab+1)^j .$$

Proof. There are $\left(\frac{a}{b}\right)^{\xi(n)} \binom{i}{j}\binom{2n-3i}{j} (ab)^{n-i-j}$ different ways to cover an n-board with i filled fences and $2n-3i$ half-squares such that exactly j of the filled fences are followed by half-squares (filled fence with a free half-square placed to its right). Select j filled fences that are followed by half-squares and j half-squares that are preceded by filled fences. In that case, there is only one way to place all the remaining filled fences, either in front of the filled fences or at the end, and all the remaining half-squares, either after the half-squares or at the beginning.

In any of these arrangements, we can choose, independently for each of the j combinations of filled fence plus half-square, whether to replace it by a bifence. This choice can be done in $(ab+1)^j$ different ways. Through this method, we ensure that every arrangement of filled fences, free half-squares, and bifences is generated exactly once. Summing over i and j gives the identity. □

Note that for $a = b = 1$, we get the following identity.

Corollary 3.3. *For any integer $n \geq 0$, we have*

$$F_{n+1}^2 = \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-3i}{j} 2^j.$$

Identity 3.4. *For any integer $n \geq 3$, we have*

$$\sum_{j=1}^{n-2} \left(\frac{b}{a}\right)^{\xi(j+1)} q_j^2 = \begin{cases} \dfrac{q_n^2 - a^2 q_{n-1}^2 - q_{n-2}^2}{2a^2}, & \textit{if } n \textit{ is even;} \\[2ex] \dfrac{q_n^2 - b^2 q_{n-1}^2 - q_{n-2}^2}{2ab}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. From Lemma 2.1 and Theorem 2.2, we get

$$q_n^2 = \begin{cases} a^2 q_{n-1}^2 + (2ab+1)\, q_{n-2}^2 + 2a^2 \sum_{j=3}^{n-1} \left(\frac{b}{a}\right)^{\xi(j+1)} q_{n-j}^2, & \textit{if } n \textit{ is even;} \\ b^2 q_{n-1}^2 + (2ab+1)\, q_{n-2}^2 + 2ab \sum_{j=3}^{n-1} \left(\frac{b}{a}\right)^{\xi(j)} q_{n-j}^2, & \textit{if } n \textit{ is odd.} \end{cases}$$

Therefore,

$$q_n^2 = \begin{cases} a^2 q_{n-1}^2 + q_{n-2}^2 + 2a^2 \sum_{j=1}^{n-2} \left(\frac{b}{a}\right)^{\xi(j+1)} q_j^2, & \textit{if } n \textit{ is even;} \\ b^2 q_{n-1}^2 + q_{n-2}^2 + 2ab \sum_{j=1}^{n-2} \left(\frac{b}{a}\right)^{\xi(j+1)} q_j^2, & \textit{if } n \textit{ is odd;} \end{cases}$$

which gives the desired result. □

Note that for $a = b = 1$, we get the following identity.

Corollary 3.5. *For any integer $n \geq 3$, we have*

$$\sum_{j=1}^{n-2} F_j^2 = \frac{F_n^2 - F_{n-1}^2 - F_{n-2}^2}{2}.$$

Identity 3.6. *For any integer $n \geq 0$, we have*

$$q_n^2 + q_{n+4}^2 = a^{2\xi(n+1)} b^{2\xi(n)} q_{n+1}^2 + 2(ab+1) q_{n+2}^2 + a^{2\xi(n+1)} b^{2\xi(n)} q_{n+3}^2.$$

Proof. For n even, by changing n to $n-2$ in (2.1) and subtracting it from (2.1), we get

$$S_n = b^2 S_{n-1} + 2(ab+1) S_{n-2} + b^2 S_{n-3} - S_{n-4} + \delta_{n,0} - \delta_{n,2}.$$

Similarly, for n odd, we get

$$S_n = a^2 S_{n-1} + 2(ab+1) S_{n-2} + a^2 S_{n-3} - S_{n-4}.$$

Therefore,

$$S_n = a^{2\xi(n)} b^{2\xi(n+1)} S_{n-1} + 2(ab+1) S_{n-2} + a^{2\xi(n)} b^{2\xi(n+1)} S_{n-3} - S_{n-4} + \delta_{n,0} - \delta_{n,2}.$$

Because $q_{n+4}^2 = S_{n+3}$, we arrive at the result. □

Note that for $a = b = 1$, we get the following identity (see [8], (37.7), p. 236)

$$F_n^2 + F_{n+4}^2 = F_{n+1}^2 + 4F_{n+2}^2 + F_{n+3}^2.$$

Identity 3.7. *For any integer $n \geq 1$, we have*

$$q_{n+2}^2 - \left(\frac{a}{b}\right)^{\xi(n+1)} (ab)^{n+1} = \sum_{k=1}^{n} \left(\frac{a}{b}\right)^{\xi(k)-\xi(n)} (ab)^{n-k} \left((2ab+1)\, q_k^2 + 2ab \sum_{i=0}^{k-1} \left(\frac{a}{b}\right)^{\xi(i)-\xi(k)} q_i^2 \right).$$

Proof. The left side of this equation represents the number of ways to tile an $(n+1)$-board containing at least one fence.

The right side is obtained by conditioning on the location of the last fence. Suppose the last fence covers cells k and $k+1$ ($k = 1, \ldots, n$), while the remaining cells to the right are covered by half squares weighted by $\left(\frac{a}{b}\right)^{\xi(k)-\xi(n)} (ab)^{n-k}$ (according to the parity of the numbers n and k). The cells k and $k+1$ can be either covered by a bifence, in which case there are q_k^2 ways to tile the left side, or they can be at the end of a mixed metatile. In the latter case, there are $2ab(q_k^2 + \left(\frac{a}{b}\right)^{\xi(k+1)-\xi(k)} q_{k-1}^2 + q_{k-2}^2 + \cdots + \left(\frac{b}{a}\right)^{\xi(k)} q_2^2 + \left(\frac{a}{b}\right)^{\xi(k+1)} q_1^2 + \left(\frac{b}{a}\right)^{\xi(k)} q_0^2)$ ways to tile the remaining cells. The result follows from considering the sum over all possible locations of the last fence. □

Note that when $a = b = 1$, we obtain the following result (see [6])

$$F_{n+2}^2 - 1 = \sum_{k=1}^{n} \left(3F_k^2 + 2 \sum_{i=0}^{k-1} F_i^2 \right).$$

Identity 3.8. *For any integer $n \geq 1$, we have*

$$q_{2n+2}^2 = a^2 + \sum_{k=1}^{n} \left(a^2 q_{2k+1}^2 + 2ab \sum_{i=1}^{2k} \left(\frac{a}{b}\right)^{\xi(i)} q_i^2 \right).$$

Proof. The left side is clearly the number of weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences tilings of a board of length $2n+1$. Because the length of the board is odd, there must be at least one half-square in each tiling. The last half-square has an odd position $2k+1$ $(k = 0, 1, \ldots, n)$, and it contributes a factor of a to the weight. Therefore, the right side can be viewed as a condition on where the last square lies. Suppose that the last half-square occupies cell $2k+1$ $(k = 0, 1, \ldots, n)$. There are $a^2 q_{2k+1}^2$ tilings, where the last half-square is part of two adjacent half-squares, or there are $2ab\left(q_{2k}^2 + \frac{a}{b}q_{2k-1}^2 + \cdots + \frac{a}{b}q_1^2\right)$ tilings, where the last half-square is part of a mixed metatile. The result follows from considering the sum of all possible locations of the last half-square. □

Note that when $a = b = 1$, we obtain the following result (see [6])

$$F_{2n+2}^2 = 1 + \sum_{k=1}^{n}\left(F_{2k+1}^2 + 2\sum_{i=1}^{2k} F_i^2\right).$$

Identity 3.9. *For $n \geq 1$, we have*

$$q_{n+1}^2 - \xi(n+1) = \left(\frac{a}{b}\right)^{\xi(n)} ab \sum_{k=1}^{n}\left(\frac{b}{a}\right)^{\xi(k+1)} (n-k+1)q_k^2.$$

Proof. The left side represents the number of ways to tile a board of length n with weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences, with at least one half-square.

The right side can be viewed as a condition on where the second-to-last half-square lies. Suppose that the second-to-last half-square occupies cell k $(k = 1, 2, \ldots, n)$. The left part represents the tilings of a length $k-1$, which can be done in q_k^2 ways. The right part represents a tiling of the remaining portion with a length of $n-k+1$, in which exactly two half-squares are used. This can be achieved in $\left(\frac{a}{b}\right)^{\xi(n)}\left(\frac{b}{a}\right)^{\xi(k+1)} ab(n-k+1)$ ways. The final result is obtained by summing over all possible values of k. □

Note that when $a = b = 1$, we obtain the following result.

Corollary 3.10. *For $n \geq 1$, we get*

$$F_{n+1}^2 - \xi(n+1) = \sum_{k=1}^{n}(n-k+1)F_k^2 = \sum_{k=1}^{n} kF_{n-k+1}^2.$$

Identity 3.11. *For $n \geq 2$ and $m \geq 1$, we have*

$$\begin{aligned} q_{m+n}^2 = &\left(\frac{b^2}{a^2}\right)^{\xi(m)\xi(n+1)} q_{m+1}^2 q_n^2 + \left(\frac{b^2}{a^2}\right)^{\xi(m+1)\xi(n)} q_m^2 q_{n-1}^2 \\ &+ 2ab\sum_{i=1}^{m} q_i^2 \sum_{j=1}^{n-1}\left(\frac{b^2}{a^2}\right)^{\xi(n+m)\xi(j+1)}\left(\frac{a}{b}\right)^{\xi(n+m-j)\xi(i)}\left(\frac{b}{a}\right)^{\xi(n+m-j+1)\xi(i+1)} q_j^2. \end{aligned}$$

Proof. The left side of this identity counts the number of ways to tile an $(m+n-1)$-board with weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences.

The right side of this identity can be viewed as a condition on the breakability of cell m. If a $(n+m-1)$-tiling is breakable at cell m, then there are q_{m+1}^2 ways to tile an m-board (left side) and $\left(\frac{b^2}{a^2}\right)^{\xi(m)\xi(n+1)} q_n^2$ ways to tile the $(n-1)$-board (right side).

If an $(n+m-1)$-tiling is unbreakable at cell m, then either there is a bifence in position $(m, m+1)$ or there is a mixed metatile in position $(m-i, m+j)$ $(i = 0, \ldots, m-1;\ j =$

$1, \dots, n-1$), weighted by $2ab\left(\frac{a}{b}\right)^{\xi(n+m-j)\xi(i)}\left(\frac{b}{a}\right)^{\xi(n+m-j+1)\xi(i+1)}$. In the case of a bifence, there are q_m^2 ways to tile a $(m-1)$-board (left side) and $\left(\frac{b^2}{a^2}\right)^{\xi(m+1)\xi(n)} q_{n-1}^2$ ways to tile the $(n-2)$-board (right side).

In the case of a mixed metatile, there are q_i^2 ways to tile an $(i-1)$-board (left side) and $\left(\frac{b^2}{a^2}\right)^{\xi(n+m)\xi(j+1)} q_j^2$ ways to tile the $(j-1)$-board (right side) for each pair (i,j). The result follows from considering the sum of all possible locations of the mixed metatile. □

Note that when $a = b = 1$, we obtain the following result.

Corollary 3.12. *For $n \geq 2$ and $m \geq 1$, we have*

$$F_{m+n}^2 = F_{m+1}^2 F_n^2 + F_m^2 F_{n-1}^2 + 2\sum_{i=1}^{m} F_i^2 \sum_{j=1}^{n-1} F_j^2.$$

Lemma 3.13. *For $n \geq 2$, let R_n be the number of ways to tile an n-board using half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences such that no free bifences occur, which satisfy the following recurrence:*

$$R_0 = 1,\ R_1 = a^2,\ \text{and } R_n = \begin{cases} \dfrac{b}{a}(ab+1)R_{n-1} + abR_{n-2}, & \text{if } n \text{ is even;} \\ \dfrac{a}{b}(ab+1)R_{n-1} + abR_{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Conditioning on the last metatile gives

$$R_n = \begin{cases} \delta_{n,0} + b^2 R_{n-1} + 2ab\sum_{j=1}^{n/2} R_{n-2j} + 2b^2\sum_{j=1}^{n/2-1} R_{n-2j-1}, & \text{if } n \text{ is even;} \\ a^2 R_{n-1} + 2ab\sum_{j=1}^{(n-1)/2} R_{n-2j} + 2a^2\sum_{j=1}^{(n-1)/2} R_{n-2j-1}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.2}$$

For n even, by changing n to $n-1$ in (3.2) and multiplying by $\dfrac{b}{a}$, we get

$$\frac{b}{a}R_{n-1} = abR_{n-2} + 2b^2\sum_{j=1}^{n/2-1} R_{n-2j-1} + 2ab\sum_{j=2}^{n/2} R_{n-2j}. \tag{3.3}$$

Subtracting (3.3) from (3.2) gives

$$R_n = \delta_{n,0} + \frac{b}{a}(ab+1)R_{n-1} + abR_{n-2}.$$

For n odd, by changing n to $n-1$ in (3.2) and multiplying by $\dfrac{a}{b}$, we get

$$\frac{a}{b}R_{n-1} = \frac{a}{b}\delta_{n,1} + abR_{n-2} + 2a^2\sum_{j=1}^{(n-1)/2} R_{n-2j-1} + 2ab\sum_{j=2}^{(n-1)/2} R_{n-2j}. \tag{3.4}$$

Subtracting (3.4) from (3.2) gives

$$R_n = -\frac{a}{b}\delta_{n,1} + \frac{a}{b}(ab+1)R_{n-1} + abR_{n-2}.$$

Therefore, $R_0 = 1$, $R_1 = a^2$, and for $n \geq 2$, we have

$$R_n = \begin{cases} \dfrac{b}{a}(ab+1)R_{n-1} + abR_{n-2}, & \text{if } n \text{ is even;} \\ \dfrac{a}{b}(ab+1)R_{n-1} + abR_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.5}$$

□

Identity 3.14. *For any integer $n \geq 2$, we have*

$$q_{n+1}^2 - R_n = \sum_{k=1}^{n-1} \left(\frac{b^2}{a^2}\right)^{\xi(n+1)\xi(k+1)} R_{n-k-1} q_k^2.$$

Proof. The left side of this identity counts the number of ways to tile an n-board that contains at least one free bifence.

The right side of this identity can be viewed as a condition on the location of the last free bifence. The total number of tilings in which the final free bifence is positioned on cells $(k, k+1)$ (where $k = 1, \dots, n-1$) can be expressed as $\left(\frac{b^2}{a^2}\right)^{\xi(n+1)\xi(k+1)} R_{n-k-1} q_k^2$. The result follows from considering the sum of all possible locations of the last free bifence. □

Note that when $a = b = 1$, we recover Identity 4.5 as presented in [6].

Lemma 3.15. *Let C_n is the number of ways to tile an n-board using weighted half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences such that no bifences occur. Then*

$$C_n = \left(\frac{a}{b}\right)^{\xi(n)} \sum_{i=0}^{\lfloor 2n/3 \rfloor} \binom{2n-2i}{i} (ab)^{n-i}.$$

Proof. From Identity 3.1, by taking $j = 0$, we get the result. □

Identity 3.16. *For any integer $n \geq 3$, we have*

$$\begin{aligned} q_{n+1}^2 - C_n = &\sum_{i=2}^{n} \left(\frac{b^2}{a^2}\right)^{\xi(n+1)\xi(i)} C_{n-i}\, q_{i-1}^2 \\ &+ \sum_{i=3}^{n} \left(\frac{b^2}{a^2}\right)^{\xi(n+1)\xi(i)} C_{n-i} \sum_{l=3}^{i} ab\,(2 - \delta_{l,3}) \left(\frac{a}{b}\right)^{\xi(i)\xi(l)} \left(\frac{b}{a}\right)^{\xi(i+1)\xi(l)} q_{i-l+1}^2. \end{aligned}$$

Proof. The left side of this identity counts the number of ways to tile an n-board that contains at least one bifence.

The right side is obtained by conditioning on the location of the last metatile containing a bifence. The last metatile can be a bifence covering cells $i-1$ and i ($i = 2, \dots, n$). In this case, the number of tilings is given by $\left(\frac{b^2}{a^2}\right)^{\xi(n+1)\xi(i)} C_{n-i} q_{i-1}^2$. Alternatively, there can be a metatile of length l covering cells $i-l+1$ to i ($i = l, \dots, n$), and in this situation, the number of tilings is $ab \left(\frac{a}{b}\right)^{\xi(i)\xi(l)} \left(\frac{b}{a}\right)^{\xi(i+1)\xi(l)} \left(\frac{b^2}{a^2}\right)^{\xi(n)\xi(i)} C_{n-i} q_{i-l+1}^2$. There is one metatile with length $l = 3$ containing a bifence and two for each $l > 3$. The result follows from the summation over i and l. □

Note that when $a = b = 1$, we obtain Identity 4.7 as given in [6].

Identity 3.17. *For any integer $n \geq 1$, we have*

$$q_{n+1}^2 = \left(\frac{a}{b}\right)^{\xi(n)} \left(\frac{b}{a}\right)^{\xi(n+1)} (ab+2)\, q_n^2 - q_{n-1}^2 + 2\left(-\frac{a}{b}\right)^{\xi(n)}.$$

Proof. We will establish a 2-to-3 correspondence between the following two sets.

- The set of all tilings of an n-board and an $(n-2)$-board.

- The set of all tilings of three $(n-1)$-boards.

We can proceed as follows:

(1) Suppose the n-board ends in a two half squares. Deleting the two half squares yields a single tiling of length $n-1$. Then there are $a^2q_n^2$ ways if n is odd and $b^2q_n^2$ ways if n is even. In other words, $\left(\frac{a}{b}\right)^{\xi(n)}\left(\frac{b}{a}\right)^{\xi(n+1)}abq_n^2$ ways.

(2) Suppose the n-board ends in a fence and contains at least one half square (i.e., neither is an all bifence tiling).
   - If the n-board ends with a bifence, determine the last half-square. If the last half-square is taken, substitute the containing filled fence with a half square. Alternatively, replace the free half-square and the following bifence to the right of it by a filled fence. This creates a tiling of the second $(n-1)$-board ending in a fence.
   - If the n-board ends with a filled fence, replace that filled fence with a half-square. This creates a tiling of the second $(n-1)$-board ending in a half-square

(3) Suppose the n-board ends in a free half square. Determine the second-to-last half-square. Subsequently, apply the same procedure used for n-boards ending in a fence to obtain the corresponding $(n-1)$-board. This creates all tilings of the third $(n-1)$-board ending in a free half-square. By applying procedure (2) to the remaining tilings of this board (i.e., those ending in a fence), we obtain the tilings of the second $(n-2)$-board containing at least one half-square.

When n is even, the n-board and the $(n-2)$-board exhibit all bifence tilings. Consequently, $q_{n+1}^2+q_{n-1}^2=\left(b^2+2\frac{b}{a}\right)q_n^2+2$. When n is odd, the all bifence tilings of the second and third $(n-1)$-boards do not correspond to any of the tilings found in the n or $(n-2)$-boards, each containing at least one half-square. Therefore, a subtraction of 2 is required. This leads to $q_{n+1}^2+q_{n-1}^2=\left(a^2+2\frac{a}{b}\right)q_n^2-2\frac{a}{b}$. □

Note that when $a=b=1$, we obtain the following result (see [6])

$$F_{n+1}^2=3F_n^2-F_{n-1}^2+2\,(-1)^n\,.$$

References

[1] N. Belaggoun and H. Belbachir, *Bi-periodic hyper-Fibonacci numbers*, Kragujevac J. Math., **49.4** (2025), 603–614.

[2] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, Dolciani Mathematical Expositions, Vol. 27, Mathematical Association of America, 2003.

[3] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's formula*, Integers, **9.6** (2009), 639–654.

[4] K. Edwards, *A Pascal-like triangle related to the Tribonacci numbers*, The Fibonacci Quarterly, **46/47.1** (2008/2009), 18–25.

[5] K. Edwards and M. A. Allen, *Strongly restricted permutations and tiling with fences*, Discret. Appl. Math., **187** (2015), 82–90.

[6] K. Edwards and M. A. Allen, *A new combinatorial interpretation of the Fibonacci numbers squared*, The Fibonacci Quarterly, **57.5** (2019), 48–53.

[7] K. Edwards and M. A. Allen, *A new combinatorial interpretation of the Fibonacci numbers squared.* Part II, The Fibonacci Quarterly, **58.2** (2020), 169–177.

[8] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[9] K. Kuhapatanakul, *Some connections between a generalized Tribonacci triangle and a generalized Fibonacci sequence*, The Fibonacci Quarterly, **50.1** (2012), 44–50.

[10] O. Yayenie, *A note on generalized Fibonacci sequences*, Appl. Math. Comput., **217.12** (2011), 5603–5611.

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