COUNTING BASE PHI REPRESENTATIONS

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Abstract. In a base phi representation, a natural number is written as a sum of powers of the golden mean φ. There are many ways to do this. How many? Even if the number of powers of φ is finite, then any number has infinitely many base phi representations. By not allowing an expansion to end with the digits 0,1,1, the number of expansions becomes finite, a solution proposed by Ron Knott. Our first result is a recursion to compute this number of expansions. This recursion is closely related to the recursion given by Neville Robbins to compute the number of Fibonacci representations of a number, also known as Fibonacci partitions. We propose another way to obtain finitely many expansions, which we call the natural base phi expansions. We prove that these are closely connected to the Fibonacci partitions.

1. Introduction

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} a_i \varphi^i,$$

where $a_i = 0$ or $a_i = 1$, and where $\varphi := (1+\sqrt{5})/2$ is the golden mean.

There are infinitely many ways to do this. When the number of powers of φ in the sum is finite, we write these *representations* (also called *expansions*) as

$$\alpha(N) = a_L a_{L-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{R+1} a_R,$$

where $a_L = a_R = 1$.

Because for all n, one has $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$, infinitely many expansions can be generated in a rather trivial way from expansions with just a few powers of φ using the replacement $1(00) \to 011$ at the right end of the expansion. So, we use Knott's truncation rule from [11],

$$a_{R+2} a_{R+1} a_R \neq 011. \tag{1.1}$$

Let $\mathrm{Tot}^\kappa(N)$ be the number of base phi expansions of the number N satisfying equation (1.1). That is,

$$\mathrm{Tot}^\kappa = 1, 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5, 10, 13, 12, 12, 13, 10, 7, 15, 18, 21, 16, 20, 20, 16, 21, 18, \dots$$ [1]

In 1957, George Bergman ([1]) proposed restrictions on the digits a_i, which entail that the representation becomes unique (proofs of this are in [15, 17]) and finite. This is generally accepted as *the* representation of the natural numbers in base phi. A natural number N is written in the Bergman representation if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

[1] In OEIS ([14]): A289749 Number of ways not ending in 011 to write n in base phi.

with digits $d_i = 0$ or $d_i = 1$, and where $d_{i+1}d_i = 11$ is not allowed. We write these representations as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

A natural number N is written in base Fibonacci if N has the form

$$N = \sum_{i=2}^{\infty} c_i F_i,$$

where $c_i = 0$ or $c_i = 1$, and $(F_i)_{i\geq 0} = 0, 1, 1, 2, 3, \dots$ are the Fibonacci numbers (defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$).

Let $\mathrm{Tot}^{\mathrm{FIB}}(N)$ be the total number of Fibonacci expansions of the number N. Then,

$$\mathrm{Tot}^{\mathrm{FIB}} = 1, 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, 3, 2, 4, 2, 3, 3, 1, 4, 3, 3, 5, \dots$$ [2]

This sequence has received a lot of attention, see e.g., the papers [2, 5, 3, 4, 8, 9, 16, 19].

In 1952, the paper [12] proposed restrictions on the digits c_i, which entail that the representation becomes unique. This is known as the Zeckendorf expansion of the natural numbers after the paper [20].

A natural number N is written in the Zeckendorf representation if N has the form

$$N = \sum_{i=2}^{\infty} e_i F_i,$$

with digits $e_i = 0$ or $e_i = 1$, and where $e_{i+1}e_i = 11$ is not allowed.

The Fibonacci representation and the base phi representation are closely related. We make a table.

Property	Fibonacci	Base phi
	$F_n: \quad n \geq 2$	$\varphi^n: \quad n$ integer
Fundamental recursion	$F_{n+1} = F_n + F_{n-1}$	$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$
Golden mean flip	$100 \to 011$	$100 \to 011$
Unique expansion	Zeckendorf	Bergman
Condition on the digits	no 11	no 11
Fundamental intervals	$[F_n, F_{n+1} - 1]$	$[L_{2n}, L_{2n+1}]$, $[L_{2n+1} + 1, L_{2n+2} - 1]$
Examples $F_5 = 5$, $L_4 = 7$	$[5, 7] = [2\,2\,1]$	$[7, 11] = [5\,8\,8\,8\,5]$
Examples $F_6 = 8$, $L_5 = 11$	$[8, 12] = [3\,2\,2\,3\,1]$	$[12, 17] = [10\,13\,12\,12\,13\,10]$

Here, the L_n are the Lucas numbers defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.
The intervals $\Lambda_{2n} = [L_{2n}, L_{2n+1}]$, $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ are called the *even* and *odd Lucas intervals*.

Replacing the digits 100 in an expansion by 011 will be called a *golden mean flip*. Our Theorem 2.1 shows that any finite base phi expansion can be obtained from the Bergman expansion by a finite number of such golden mean flips. There is a special case that needs attention, which we illustrate with an example. Let $N = 4$. Then, $\beta(4) = 101 \cdot 01$. Applying the golden mean flip at the right gives the expansion $101 \cdot 0011$, which is not an allowed

[2] In OEIS ([14]): A000119 Number of representations of n as a sum of distinct Fibonacci numbers.

expansion. However, if we apply a second golden mean flip, we can obtain $100 \cdot 1111$, which *is* an allowed expansion. We call this operation a *double golden mean flip.*

In Section 2, we determine a formula for $\mathrm{Tot}^{\kappa}(N)$. In Section 3, we give simple formulas for $N = F_n$, and for $N = L_n$. In Section 4, we introduce a new way to count expansions, by defining *natural expansions*, and give a formula for $\mathrm{Tot}^{\nu}(N)$, the number of natural base phi expansions of N. We also show that $(\mathrm{Tot}^{\nu}(N))$ is a subsequence of the sequence of total numbers of Fibonacci representations. Section 5 gives important information on the different behavior of phi expansions on the odd and the even Lucas intervals.

We finally mention that our results have been recently reproved by Shallit in the paper [18], using the Walnut software.

2. A Recursive Formula for the Number of Knott Expansions

In this section, we determine a formula for $\mathrm{Tot}^{\kappa}(N)$ for each natural number N.

The emphasis will be on the manipulation of 0-1-words, not on base phi expansions of numbers.

Let $\alpha(N) = a_La_{L-1}\dots a_1a_0 \cdot a_{-1}a_{-2}\dots a_{R+1}a_R$ be a base phi representation of N. By removing the radix point, we obtain a 0-1-word $A(N) := a_La_{L-1}\dots a_1a_0a_{-1}a_{-2}\dots a_{R+1}a_R$. Such a word will be called a *base phi word.* Similarly, the *Bergman word* $B(N)$ will be the unique 0-1-word obtained by removing the radix point from the Bergman expansion $\beta(N)$ of N.

We keep the indexing with L and R, and in decreasing order, to facilitate the connection with base phi expansions.

We apply golden mean flips to these 0-1-words. Such a golden mean flip may change the length of the word, and the property $a_L = a_R = 1$. To cope with this, it is useful to consider the three *companion words* $0A(N)$, $A(N)00$, and $0A(N)00$ of a base phi word $A(N)$. In particular, we will identify the Bergman 0-1-word $B(N)$ with its three companion words in the proof of Theorem 2.1.

We map any base phi word $A(N) = a_La_{L-1}\dots a_{R+1}a_R$ with $a_{i+1}a_ia_{i-1} = 100$ for some i with $R+1 \le i \le L-1$ to another 0-1-word, by the map

$$T_i : \dots a_{i+1}a_ia_{i-1}\dots \to \dots [a_{i+1}-1][a_i+1][a_{i-1}+1]\dots.$$

This is the golden mean flip. We also allow T_{R-1} on the companion word $A(N)00$ of $A(N)$.

The map T_i has an inverse denoted U_i for $R+1 \le i \le L-1$ given by

$$U_i : \dots a_{i+1}a_ia_{i-1}\dots \to \dots [a_{i+1}+1][a_i-1][a_{i-1}-1]\dots,$$

as soon as $a_{i+1}a_ia_{i-1} = 011$. We also allow U_L on the companion word $0A(N)$ of $A(N)$.

We call the maps U_i *reverse golden mean flips.*

Example 1. *Suppose* $N = 11$. *Then* $\beta(N) = 10101 \cdot 0101$, *so* $B(N) = 101010101$. *Let* $\alpha(N) = 10101 \cdot 001111$, *so* $L = 4$, $R = -6$, *and* $A(N) = 10101001111$.

Then $U_{-3}(A(N)) = 10101010011$, *and* $U_{-5}U_{-3}(A(N)) = 10101010100$, *which is a companion of the Bergman word* $B(N)$.

Theorem 2.1. *Any finite base phi expansion* $\alpha(N)$ *with digits 0 and 1 of a natural number* N *can be obtained from the Bergman expansion* $\beta(N)$ *of* N *by a finite number of applications of the golden mean flip.*

Proof. We prove this by showing that any base phi word $A(N)$ will be mapped to the Bergman word $B(N)$ or one of its companions by a finite number of applications of the reverse golden mean flip. Let $A(N) = a_L a_{L-1} \dots a_{R+1} a_R$ be a base phi word associated to the expansion of N with digits 0 and 1. When 11 does not occur in $A(N)$, then $A(N) = B(N)$ or one of its companions, and there is nothing to do. Otherwise, let $m := \max\{i : a_i a_{i-1} = 11\}$. First, suppose $m \leq L-2$. Then, by the definition of m, we have $a_{i+1} = 0$ if $i = m$. So, for the two possibilities $a_{i+2} = 0$ and $a_{i+2} = 1$,

$$U_i(\dots 0 a_{i+1} a_i a_{i-1} \dots) = U_i(\dots 0011 \dots) = \dots 0100 \dots,$$
$$U_i(\dots 1 a_{i+1} a_i a_{i-1} \dots) = U_i(\dots 1011 \dots) = \dots 1100 \dots.$$

Note that in the first case, the total number of 11 occurring in $A(N)$ has decreased by 1, and in the second case, it remained constant. However, in the second case, the m of $U_i(A(N))$ has increased by 2. If we keep iterating the reverse golden mean flip on the left most occurrence of 11, then 0011 will occur, or if not, then $A(N) = 1101 \dots$. This is the case $m = L$, where there *is* a decrease in the number of 11, because $U_L(0A(N)\dots) = 10001\dots$. The conclusion in all cases is that the number of 11 will decrease by at least 1 after a finite number of applications of the reverse golden mean flip. So after a finite number of applications of the reverse golden mean flip, we reach a 0-1-word with no occurrences of 11. By definition, this is the Bergman word $B(N)$ or one of its companions.

The case $m = L$ has already been considered above; the case $m = L-1$ corresponds to $A(N) = 011\dots$, where an application of the reverse golden mean flip leads also to a decrease in the number of 11. □

Our proof for Tot^κ resembles the work of Neville Robbins [16] on Fibonacci representations, but we have to incorporate the double golden mean flip defined in the Introduction. It then will appear that the two recursions for Fibonacci representations and golden mean (Knott) representations are the same, but that there is a difference in the initial conditions.

Let $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$. As before, by removing the radix point, we obtain a 0-1-word $B(N) = d_L d_{L-1} \dots d_1 d_0 d_{-1} d_{-2} \dots d_{R+1} d_R$. Let us denote

$$r(B(N)) := \mathrm{Tot}^\kappa(N).$$

Remark 2.2. *Before we continue with the determination of $r(B(N))$, we remark that in general, the representations that we obtain by golden mean flips are not representations of a natural number—not for any choice of the radix point. An example is $w = 100001$, which represents $\varphi^5 + 1$, and its multiplications by φ and φ^{-1}. Nevertheless, these words represent numbers $a + b\varphi$ with nonnegative natural numbers a and b in the ring $\mathbb{Z}(\varphi)$.*

For example, $w = 100001$ represents $5\varphi + 4$, which is a direct consequence of the relation $\varphi^2 = \varphi + 1$. This is the justification for continuing with the terminology of representations.

A 0-1-word that plays an important role in the analysis that follows is the word 10^s for $s > 1$. Although 10^s is not a base phi representation, it is convenient to call the word 10^s and its golden mean flip iterates representations of 10^s. Let $q(10^s)$ be the number of such representations. Then,

$$q(10^s) = \begin{cases} \frac{1}{2}s + 1, & \text{if } s \text{ is even;} \\ \frac{1}{2}(s+1), & \text{if } s \text{ is odd.} \end{cases} \tag{2.1}$$

This follows by making golden mean flips from left to right.

Suppose a 0-1-word is of the form 10^s1. Then we have

$$r(10^s1) = \begin{cases} \frac{1}{2}s + 1, & \text{if } s \text{ is even;} \\ \frac{1}{2}(s+1) + 1, & \text{if } s \text{ is odd.} \end{cases} \tag{2.2}$$

This follows because 10^s1 has the same number of representations $q(10^s)$ as 10^s when s is even, but there is one extra representation generated by the double golden mean flip when s is odd.

Suppose the Bergman representation $\beta(N)$ of a number N contains $n+1$ ones. Then, we can write for some numbers s_1, s_2, …, s_n,

$$B(N) = 10^{s_n} \dots 10^{s_2}\, 10^{s_1}\, 1.$$

We start with the case $n = 2$, so

$$B(N) = 10^{s_2}10^{s_1}1.$$

Let us call $I_2 := 10^{s_2}$ the *initial segment* of $B(N)$, and $T_1 := 10^{s_1}1$ the *terminal segment* of $B(N)$.
We want to deduce $r(B(N)) = r(I_2T_1)$ from the number of representations $q(I_2)$ and $r(T_1)$. There are two cases to consider.

Type 1: Arbitrary combinations of representations of I_2 and T_1.
Type 2: Arbitrary combinations of representations of I_2 and T_1 *plus* an 'overlap' combination.

Type 1 typically occurs if s_2 is even. For example for the case $s_2 = 4$, we have the three representations 10000, 01100, 01011. Note that in general, these representations always end in 00 or 11.
So for Type 1, one has simply

$$r(B(N)) = r(I_2T_1) = q(I_2)r(T_1). \tag{2.3}$$

But for s_2 odd, for example when $s_2 = 5$, then 100000, 011000, 010110 are the three representations of I_2. Note that in general, these representations always end in 00 or 10.
So if a representation of the segment I_2 is of the form $w10$, and a representation of T_1 is of the form $0v$, then the representation $w10\,0v$ of I_2T_1 generates an 'overlap' representation $w01\,1v$ via the golden mean flip.
Obviously, it is true in general that an I_2 word with s_2 odd will have exactly one representation that ends in 10. It is also important to note that there is no representation that ends in 01. Therefore, if $r^{(i)}(T_1)$ denotes the number of representations of T_1 starting with i for $i = 0, 1$, then we obtain for Type 2,

$$r(B(N)) = r(I_2T_1) = q(I_2)r(T_1) + r^{(0)}(T_1). \tag{2.4}$$

When we combine equation (2.4), the trivial equation $r^{(0)}(T_1) + r^{(1)}(T_1) = r(T_1)$, and the fact that the segment $T_1 = 10^{s_1}1$ has just one representation that starts with a 1, we obtain

$$r(B(N)) = q(I_2)r(T_1)+r(T_1)-r^{(1)}(T_1) = r(T_1)[q(I_2)+1]-r^{(1)}(T_1) = r(T_1)[q(I_2)+1]-1. \tag{2.5}$$

We continue with the case $n = 3$, so

$$B(N) = 10^{s_3}10^{s_2}10^{s_1}1.$$

Now $I_3 := 10^{s_3}$ is the *initial segment*, and $T_2 := 10^{s_2}10^{s_1}\,1$ is the *terminal segment*.
As before, there are two cases to consider to compute $r(B(N)) = r(I_3T_2)$.

Type 1: Arbitrary combinations of representations of I_3 and T_2.
Type 2: Arbitrary combinations of representations of I_3 and T_2 *plus* an 'overlap' combination.

For Type 1, one has simply

$$r(B(N)) = r(I_3T_2) = q(I_3)r(T_2). \tag{2.6}$$

For Type 2, one has

$$r(B(N)) = r(I_3T_2) = q(I_3)r(T_2) + r^{(0)}(T_2). \tag{2.7}$$

Next, we split $T_2 = I_2T_1$, where $I_2 := 10^{s_2}$. Then we have, because I_2 has just one representation that starts with a 1, that $r^{(1)}(T_2) = r(T_1)$. It thus follows from equation (2.7) and $r^{(0)}(T_2) + r^{(1)}(T_2) = r(T_2)$ that

$$r(B(N)) = q(I_3)r(T_2)+r(T_2)-r^{(1)}(T_2) = r(T_2)[q(I_3)+1]-r^{(1)}(T_2) = r(T_2)[q(I_3)+1]-r(T_1). \tag{2.8}$$

For general n, we split $B(N) = 10^{s_n} \dots 10^{s_2}\, 10^{s_1}1$ in an initial segment $I_n = 10^{s_n}$ and a terminal segment $T_{n-1} = 10^{s_{n-1}} \dots 10^{s_1}1$. We then find, in the same way as for the case $n = 3$, that for s_n even,

$$r(T_n) = r(B(N)) = q(I_n)r(T_{n-1}), \tag{2.9}$$

and for s_n odd,

$$r(T_n) = r(B(N)) = r(T_{n-1})[q(I_n) + 1] - r(T_{n-2}). \tag{2.10}$$

Defining $r_n := r(B(N))$, $r_k := r(T_k)$ for $k = 1, \dots, n-1$ and $r_0 = 1$ (cf. equation (2.5)), we have obtained a recursion that computes $r(B(N))$.

Theorem 2.3. *For any integer $N \geq 2$, let the Bergman expansion $\beta(N) = d_L \dots d_0 \cdot d_{-1} \dots d_R$ of N have $n+1$ digits 1. Let $\mathrm{Tot}^{\kappa}(N) = r_n$ be the number of Knott representations of N. Define the initial conditions: $r_0 = 1$ and $r_1 = \frac{1}{2}s_1 + 1$ if s_1 is even, $r_1 = \frac{1}{2}(s_1+1) + 1$ if s_1 is odd. Then for $n \geq 2$,*

$$r_n = \begin{cases} [\frac{1}{2}s_n+1]\, r_{n-1}, & \text{if } s_n \text{ is even;} \\ [\frac{1}{2}(s_n+1) + 1]r_{n-1} - r_{n-2}, & \text{if } s_n \text{ is odd.} \end{cases}$$

The initial condition for r_1 (given by equation (2.2)) is different from the Fibonacci case; if s_1 is odd, then the base phi expansion has an extra representation that is generated by the double golden mean flip.

3. Expansions of the Fibonacci Numbers and the Lucas Numbers

Let $(F_n) = 0, 1, 1, 2, 3, 5, \dots$ be the Fibonacci numbers. We will determine the number of Knott representations of these numbers. First, we have to find a formula for the Bergman expansions of the Fibonacci numbers. Let $B(N)$ be $\beta(N)$ without the radix point in the expansion.

Proposition 3.1. *For $n \geq 1$, one has*

a) $B(F_{2n}) = (1000)^{n-1}1$.
b) $B(F_{2n+1}) = (1000)^{n-1}1001$.

Proof. This will be proved by induction. We note that $\beta(F_2) = \beta(1) = 1$, $\beta(F_3) = \beta(2) = 10 \cdot 01$, $\beta(F_4) = \beta(3) = 100 \cdot 01$, and $\beta(F_5) = \beta(5) = 1000 \cdot 1001$. So the statements hold for $n = 1, 2$.

The induction step is based on adding $\beta(F_{m-1})$ and $\beta(F_m)$ for all $m \geq 4$. We therefore need the position of the radix point in these expansions. This is determined by giving $L(F_m)$, which we claim is equal to $L(F_m) = m - 2$. The validity of this claim can be readily seen from the expansions for $m = 4, 5$ above, and will follow for $m \geq 6$ directly from the induction proof that we give below.

We illustrate the induction step by giving the case $n = 3$. Because $F_6 = F_4 + F_5$, we have

$$\begin{aligned} \beta(F_4) &= \quad 100 \cdot 01, \\ \beta(F_5) &= \ 1000 \cdot 1001, \\ \beta(F_4) + \beta(F_5) &= \ 1100 \cdot 1101, \\ \beta(F_4) + \beta(F_5) &= 10001 \cdot 0001 \Rightarrow B(F_6) = (1000)^2 1. \end{aligned}$$

Here, we applied the reverse golden mean flip twice in the last step, and because the last expansion does not have any 11, we could conclude that $\beta(F_6) = 10001 \cdot 0001$. Next, we show what happens at $F_7 = F_5 + F_6$.

$$\begin{aligned} \beta(F_5) &= \quad 1000 \cdot 1001, \\ \beta(F_6) &= \ 10001 \cdot 0001, \\ \beta(F_5) + \beta(F_6) &= \ 11001 \cdot 1002 \Rightarrow \beta(F_5) + \beta(F_6) = 11001 \cdot 101001, \\ \beta(F_5) + \beta(F_6) &= 100010 \cdot 001001 \Rightarrow B(F_7) = (1000)^2 1001. \end{aligned}$$

Here, we used a shifted version of $\beta(2) = 10 \cdot 01$, and we applied the reverse golden mean flip twice in the last step. Because the last expansion does not have any 11, we can conclude that $\beta(F_7) = 100010 \cdot 001001$.

Suppose the formulas hold for the numbers $1, \ldots, 2n-1$. Then, $\beta(F_{2n})$ is determined by first obtaining a base phi representation $\alpha(F_{2n})$ of F_{2n} by way of

$$\alpha(F_{2n}) := \beta(F_{2n-2}) + \beta(F_{2n-1}).$$

We see that the corresponding 0-1 base phi word is equal to $A(F_{2n}) = (1100)^{n-2}1101$.

Next, $n-1$ reverse golden mean flips transform $A(F_{2n})$ to another base phi word $A'(F_{2n}) = (1000)^{n-1}1$. But then, the Bergman word $B(F_{2n}) = A'(F_{2n}) = (1000)^{n-1}1$ because 11 does not occur in $A'(F_{2n})$.

Then, $\beta(F_{2n+1})$ is determined by first obtaining a base phi representation $\alpha(F_{2n+1})$ of F_{2n+1} by way of

$$\alpha(F_{2n+1}) := \beta(F_{2n-1}) + \beta(F_{2n}).$$

This time, the addition gives the word $(1100)^{n-1}2$, which represents F_{2n+1}, but is not a 0-1-word. We get rid of the 2 by replacing 02 by 1001 in the companion word $(0110)^{n-1}02$ of this word, resulting in the companion base phi word $0A(F_{2n+1}) := (0110)^{n-1}1001$.

Next, $n-1$ reverse golden mean flips transform $0A(F_{2n+1})$ to a base phi word $A'(F_{2n+1}) = (1000)^{n-1}1001$. But then the Bergman word $B(F_{2n+1}) = A'(F_{2n+1}) = (1000)^{n-1}1001$, because 11 does not occur in $A'(F_{2n+1})$.

This finishes the induction proof. □

Theorem 3.2. *For all $n \geq 1$, one has $\mathrm{Tot}^{\kappa}(F_n) = F_n$.*

Proof. It can be checked that the proposition holds for $n = 1$ and $n = 2$. So, let $n \geq 3$. From Proposition 3.1, the number of ones in $\beta(F_n)$ is $p+2$, where p is defined by $p+2 = (n+1)/2$ if n is odd, and $p+2 = n/2$ if n is even. Also, $\beta(F_n) = 10^{s_{p+1}} \dots 10^{s_k} \dots 10^{s_1} 1$, with $s_k = 3$ for $k = 2, \dots, p+1$, and $s_1 = 2$ for n odd, $s_1 = 3$ for n even.

We apply Theorem 2.3. This yields that $\mathrm{Tot}^\kappa(F_n) = r_{p+1}$, the number of Knott representations of the Bergman representation of F_n satisfies

$$r_{p+1} = 3r_p - r_{p-1}.$$

Here, the initial conditions are $r_0 = 1$, $r_1 = s_1/2+1 = 2$ for n even, and $r_1 = (s_1+1)/2+1 = 3$ for n odd.
The same recurrence relation holds for the subsequences of even and odd Fibonacci numbers,

$$F_{n+1} = F_n + F_{n-1} = 2F_{n-1} + F_{n-2} = 3F_{n-1} - F_{n-1} + F_{n-2} = 3F_{n-1} - F_{n-3}. \tag{3.1}$$

(I) Suppose $n = 2m+1$ is odd. Then $p = m-1$, so $\mathrm{Tot}^\kappa(F_{2m+1}) = r_m$.
We claim that $r_m = F_{2m+1}$ for all $m \geq 0$.
For $m = 0$, we have $r_0 = 1 = F_1$, and for $m = 1$, we have $r_1 = 2 = F_3$.
For $m \geq 2$,

$$r_m = 3r_{m-1} - r_{m-2} = 3F_{2m-1} - F_{2m-3} = F_{2m+1},$$

by the induction hypothesis and equation (3.1).

(II) Suppose $n = 2m+2$ is even. Then $p = m-1$, so $\mathrm{Tot}^\kappa(F_{2m+2}) = r_m$.
We claim that $r_m = F_{2m+2}$ for all $m \geq 0$.
For $m = 0$, we have $r_0 = 1 = F_2$, and for $m = 1$, we have $r_1 = 3 = F_4$.
For $m \geq 2$,

$$r_m = 3r_{m-1} - r_{m-2} = 3F_{2m} - F_{2m-2} = F_{2m+2},$$

by the induction hypothesis and equation (3.1).
Combining (I) and (II) yields the conclusion, $\mathrm{Tot}^\kappa(F_n) = F_n$ for all $n \geq 1$. □

At the Fibonacci numbers, the total number of expansions is very large, but here, we show that it is rather small at the Lucas numbers (L_n).

Theorem 3.3. *For all* $n \geq 1$*, one has* $\mathrm{Tot}^\kappa(L_{2n}) = \mathrm{Tot}^\kappa(L_{2n+1}) = 2n+1$.

Proof. The Lucas numbers have simple representations: $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1$, $\beta(L_{2n+1}) = 1(01)^n \cdot (01)^n$. For a proof, see Example 3 in Section 4.

So, the representation of L_{2n} has only two ones. It follows, from Theorem 2.3, that $\mathrm{Tot}^\kappa(L_{2n}) = r_1 = (s_1+1)/2+1 = 2n+1$, because $s_1 = 4n-1$ is odd.

The representation of L_{2n+1} has $2n+1$ ones, and each s_k of the blocks 10^{s_k} is equal to 1, which is odd. It follows, from Theorem 2.3, that $\mathrm{Tot}^\kappa(L_{2n+1}) = r_n = 2r_{n-1} - r_{n-2}$. And, induction gives that $r_n = 2(2n-1) - (2n-3) = 2n+1$. □

4. Natural Base Phi Expansions

A consequence of the application of the double golden mean flip is that the length of the negative part of the Knott expansions may take two different values.

To obtain what we will call the *natural* expansions, let us delete all expansions that have a length of the negative part that is not equal to the length of the negative part of the Bergman expansion.

For example, in the case $N = 4$, Knott proposes the three expansions $101 \cdot 01$, $100 \cdot 1111$, and 11.1111. However, there is only one natural expansion, the Bergman expansion $101 \cdot 01$.

Let $\mathrm{Tot}^{\nu}(N)$ denote the number of natural base phi expansions. Then we have

$$(\mathrm{Tot}^{\nu}(N)) = 1,1,2,2,1,5,5,4,5,4,3,1,10,13,12,12,13,10,6,11,12,\dots$$

instead of

$$(\mathrm{Tot}^{\kappa}(N)) = 1,1,2,3,3,5,5,5,8,8,8,5,10,13,12,12,13,10,7,15,18,\dots$$

The number of natural base phi expansions can be determined in a way that is similar to the Knott expansion case.

Theorem 4.1. *For a natural number N, let the Bergman expansion of N have $n+1$ digits 1. Suppose $\beta(N) = 10^{s_n}\dots 10^{s_1}1$. Let $\mathrm{Tot}^{\nu}(N) = r_n$ be the number of natural base phi representations of N. Define the initial conditions: $r_0 = 1$ and $r_1 = \frac{1}{2}s_1 + 1$ if s_1 is even, $r_1 = \frac{1}{2}(s_1+1)$ if s_1 is odd. Then, for $n \geq 2$,*

$$r_n = \begin{cases} [\frac{1}{2}s_n+1]\, r_{n-1}, & \text{if } s_n \text{ is even;} \\ [\frac{1}{2}(s_n+1)+1] r_{n-1} - r_{n-2}, & \text{if } s_n \text{ is odd.} \end{cases}$$

Proof. This follows directly from Theorem 2.3 and its proof. The only difference between the process of generating all Knott expansions and all natural expansions is the double golden mean flip, which is performed in the Knott expansion at the segment $10^{s_1}1$, and only when s_1 is odd. So, $\mathrm{Tot}^{\nu}(N) = r_n$ satisfies the same recursion as $\mathrm{Tot}^{\mathrm{FIB}}(N)$, except that $r_1 = \frac{1}{2}(s_1+1)+1$ has to be replaced by $r_1 = \frac{1}{2}(s_1+1)$ in the case that s_1 is odd. □

We will determine the total number of natural expansions of the Fibonacci numbers. First, we present a lemma that emphasizes the interconnection between the Fibonacci and the Lucas numbers. Recall the even and odd Lucas intervals, $\Lambda_{2n} = [L_{2n},\, L_{2n+1}]$, $\Lambda_{2n+1} = [L_{2n+1} + 1,\, L_{2n+2} - 1]$ (cf. [6]).

Lemma 4.2. *For all $n = 1,2,\dots,$ one has $F_{2n+2} \in \Lambda_{2n}$, $F_{2n+3} \in \Lambda_{2n+1}$.*

Proof. By induction. For $n = 1$, we have $F_4 = 3 \in \Lambda_2 = [3,4]$, and $F_5 = 5 \in \Lambda_3 = [5,6]$.
For $n = 2$, we have $F_6 = 8 \in \Lambda_4 = [7,11]$, and $F_7 = 13 \in \Lambda_5 = [12,17]$.

Suppose the statement of the lemma has been proved for F_{2n+1} and F_{2n+2}. So. we know

$$\begin{aligned} F_{2n+1} &\in [L_{2n-1}+1,\, L_{2n}-1] = \Lambda_{2n-1}, \\ F_{2n+2} &\in [L_{2n},\, L_{2n+1}] \qquad\quad = \Lambda_{2n}. \end{aligned}$$

Adding the numbers in these two equations vertically, we obtain

$$F_{2n+3} \in [L_{2n+1}+1,\, L_{2n+2}-1] = \Lambda_{2n+1}.$$

We can then write

$$\begin{aligned} F_{2n+2} &\in [L_{2n},\, L_{2n+1}] \qquad\quad = \Lambda_{2n}, \\ F_{2n+3} &\in [L_{2n+1}+1,\, L_{2n+2}-1] = \Lambda_{2n+1}. \end{aligned}$$

This time, adding gives

$$F_{2n+4} \in [L_{2n+2}+1,\, L_{2n+3}-1] \subset [L_{2n+2},\, L_{2n+3}] = \Lambda_{2n+2}.$$

□

Theorem 4.3. *For all $n = 0,1,2,\dots,$ one has $\mathrm{Tot}^{\nu}(F_{2n+2}) = F_{2n+1}$ and $\mathrm{Tot}^{\nu}(F_{2n+3}) = F_{2n+3}$.*

Proof. We use the result from Proposition 5.1, which gives that for all N from Λ_{2n+1}, if $\beta(N) = ...10^{s_1}1$, then s_1 is even. So for all N from Λ_{2n+1}, we have that the total number of natural expansions is equal to the total number of Knott expansions. In particular, we obtain from Lemma 4.2, using Theorem 3.2, that

$$\mathrm{Tot}^{\nu}(F_{2n+3}) = \mathrm{Tot}^{\kappa}(F_{2n+3}) = F_{2n+3}.$$

From Proposition 3.1, we have that $B(F_{2n+2}) = (1000)^n 1$. Therefore, Theorem 4.1 gives that (r_n) satisfies the recurrence relation $r_n = 3r_{n-1} - r_{n-2}$, with $r_1 = \frac{1}{2}(3+1) = 2 = F_3$. This is the recurrence relation for the Fibonacci numbers with odd indices, cf. equation (3.1). Therefore, $\mathrm{Tot}^{\nu}(F_{2n+2}) = r_n = F_{2n+1}$. □

There is a direct connection between the total number of natural expansions and the total number of Fibonacci expansions.

Theorem 4.4. *For every $N > 3$, let $\beta(N) = d_{L(N)} \dots d_{R(N)}$ be the Bergman expansion of N. Then*

$$\mathrm{Tot}^{\nu}(N) = \mathrm{Tot}^{\mathrm{FIB}}(F_{-R(N)+2}\, N).$$

Proof. Suppose that $\beta(N) = d_L \dots d_R$, so $N = \sum_R^L d_i \varphi^i$.
Multiplying by φ^{-R+2}, we get

$$\varphi^{-R+2} N = \sum_{i=R}^{L} d_i \varphi^{i-R+2} = \sum_{j=2}^{L-R+2} d_{j+R-2}\varphi^j = \sum_{j=2}^{L-R+2} e_j \varphi^j,$$

where we substituted $j = i - R + 2$, and defined $e_j := d_{j+R-2}$.

Next, we use the well known equation $\varphi^j = F_j\varphi + F_{j-1}$ to obtain

$$[F_{-R+2}\varphi + F_{-R+1}]N = \sum_{j=2}^{L-R+2} e_j[F_j\varphi + F_{j-1}].$$

This implies that

$$F_{-R+2}N = \sum_{j=2}^{L-R+2} e_j F_j.$$

We conclude that the number $F_{-R+2}N$ has a Zeckendorf expansion given by the sum on the right side.

But the manipulations above can be made for any 0-1-word of length $L-R+1$, so the golden mean flips of $d_L \dots d_R$ are in 1-to-1 correspondence with golden mean flips of $e_2 \dots e_{L-R+2}$. This implies that $\mathrm{Tot}^{\nu}(N) = \mathrm{Tot}^{\mathrm{FIB}}(F_{-R(N)+2}N)$. □

Example 2. *The Bergman expansion of* 4 *is* 101·01*, and $F_4 = 3$. So* $\mathrm{Tot}^{\nu}(4) = \mathrm{Tot}^{\mathrm{FIB}}(12) = 1$.

Example 3. *The Bergman expansion of* 14 *is* $100100 \cdot 001001$*, and $F_8 = 21$. So,* $\mathrm{Tot}^{\nu}(14) = \mathrm{Tot}^{\mathrm{FIB}}(294) = 12$.

Example 4. *Consider the Lucas numbers. From* $L_{2n} = \varphi^{2n} + \varphi^{-2n}$*, and* $L_{2n+1} = L_{2n} + L_{2n-1}$*,*

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

We read off that $R(L_{2n}) = -2n$*,* $R(L_{2n+1}) = -2n$*.*

It is also clear that $\mathrm{Tot}^{\nu}(L_{2n}) = 2n$*, and* $\mathrm{Tot}^{\nu}(L_{2n+1}) = 1$*.*

So, Theorem 4.4 gives the total number of Fibonacci representations of $F_{2n+2}L_{2n}$ *and* $F_{2n+2}L_{2n+1}$*, i.e.,*

$\mathrm{Tot}^{\mathrm{FIB}}(F_{2n+2}L_{2n}) = 2n$, $\mathrm{Tot}^{\mathrm{FIB}}(F_{2n+2}L_{2n+1}) = 1$ *for all* $n \geq 1$*.*

We find in [14]*, from Miklos Kristof, the following result.*

Let $L(n) = A000032(n)$ *= Lucas numbers. Then, for* $a \leq b$ *and odd* b*,* $F(a+b) - F(a-b) = F(a) * L(b)$*.*

So, $F_{2n+2}L_{2n+1} = F_{4n+3} - F_1 = F_{4n+3} - 1$*. But,* $\mathrm{Tot}^{\mathrm{FIB}}(F_n - 1) = 1$ *is a well-known formula.*

5. Comparing Knott Expansions and Natural Expansions

It is not hard to see that the double golden mean flip—in general, combined with more golden mean flips—can be applied if and only if the expansion ends in 10^s1, where s is odd. So, the difference between the Knott expansions and the natural expansions is made more explicit by part a) of the following result.

Proposition 5.1.
a) *A number* $N \geq 2$ *is in* Λ_{2n} *for some integer* n *if and only if* $\beta(N) = ...10^s1$*, where* s *is odd, and* $N \geq 2$ *is in* Λ_{2n+1} *for some integer* n *if and only if* $\beta(N) = ...10^s1$*, where* s *is even.*
b) *Let* $\beta(N) = L(N)...R(N)$*. A number* N *in* Λ_{2n} *has* $-R(N) = 2n$*, a number* N *in* Λ_{2n+1} *has* $-R(N) = 2n + 2$*.*

Proposition 5.1 will be proved by induction. Thus, we need recursions to let the proof work. These are given in the paper [7], from which we repeat the following.

To obtain recursive relations, the interval $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ has to be divided into three subintervals. These three intervals are

$$\begin{aligned} I_n &:= [L_{2n+1} + 1,\ L_{2n+1} + L_{2n-2} - 1], \\ J_n &:= [L_{2n+1} + L_{2n-2},\ L_{2n+1} + L_{2n-1}], \\ K_n &:= [L_{2n+1} + L_{2n-1} + 1,\ L_{2n+2} - 1]. \end{aligned}$$

It will be convenient to extend the monoid of words of 0's and 1's to the corresponding free group. So, for example, $1000(10)^{-1}1001 = 100001$.

Theorem 5.2. [Recursive Structure Theorem, [7]**]**
I) *For all* $n \geq 1$ *and* $k = 0, \ldots, L_{2n-1}$*, one has*

$$\beta(L_{2n} + k) = \beta(L_{2n}) + \beta(k) = 10 \ldots 0\, \beta(k)\, 0 \ldots 01.$$

II) *For all* $n \geq 2$ *and* $k = 1, \ldots, L_{2n-2} - 1$*,*

$$\begin{aligned} I_n: &\quad \beta(L_{2n+1} + k) = 1000(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}1001, \\ K_n: &\quad \beta(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}0001. \end{aligned}$$

Moreover, for all $n \geq 2$ *and* $k = 0, \ldots, L_{2n-3}$*,*

$$J_n: \quad \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

Proof of Proposition 5.1. To start the induction, we note that

$$\Lambda_2 = [3,4]; \quad \beta(3) = 100 \cdot 01,\ \beta(4) = 101 \cdot 01,$$
$$\Lambda_3 = [5,6]; \quad \beta(5) = 1000 \cdot 1001,\ \beta(6) = 1010 \cdot 0001.$$

For the even intervals, we have that $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1$, so the expansion of the first element ends in 10^s1, where s is odd. Note also that $R(L_{2n}) = 2n$, and this property will hold for all $L_{2n} + k,\ k = 0, \dots, L_{2n-1}$ because the sum $\beta(L_{2n}) + \beta(k)$ in I) does not change the length of the negative part. Moreover, because the length of the negative part of each $\beta(k)$ in the sum $\beta(L_{2n}) + \beta(k)$ is even (by the induction hypothesis for part b)), the expansion must end in 10^s1 with s odd, simply because the difference of two even numbers is even.

For the odd intervals, we have to consider the three cases from II).

For I_n, we know that $\beta(L_{2n-1} + k)$ ends in 01, so $\beta(L_{2n+1} + k)$ ends in 1001. For part b), the length of the negative part is increased by 2.

For K_n, $L_{2n-1} + k$ is from an odd interval, so the expansion ends in $10^{2t}1$ from some $t > 0$. But then, the expansion of $L_{2n+1}+L_{2n-1}+k$ ends in $10^{2t}1\,(01)^{-1}0001 = 10^{2t-1}0001 = 10^{2t+2}1$. For part b), the length of the negative part is increased by 2.

For J_n, $\beta(L_{2n+1} + L_{2n-2} + k)$ ends in 1001. For part b), the length of the negative part is $2n - 2 + 4 = 2n + 2$. □

Acknowledgement

We are grateful to an anonymous referee for the careful reading of our manuscript, and for proposing many improvements.

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MSC2020: 11B39, 68R15, 11B37

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