CATALAN IDENTITIES FOR GENERALIZED FIBONACCI POLYNOMIALS

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Abstract. This paper extends two Catalan identities, originally formulated for the Fibonacci and Lucas numbers, to polynomial sequences of the second order, which have a Binet formula similar to that of the Fibonacci and Lucas numbers. These polynomial sequences are classified as of Fibonacci type and Lucas type. As a result of this generalization, Catalan identities are obtained for a range of polynomial sequences, such as Pell, Pell-Lucas, Fermat, Fermat-Lucas, both types of Chebyshev polynomials, Jacobsthal, Jacobsthal-Lucas, and both types of Morgan-Voyce polynomials.

Furthermore, we use generating functions and the Wilf-Zeilberger algorithm to derive a general expression for Catalan identities and other combinatorial identities.

1. Introduction

The Binet formula is a fundamental tool in the study of linear recursive sequences, and the Binet formulas for Fibonacci and Lucas numbers are well known. In this context, a second-order polynomial sequence is said to be of Fibonacci type (Lucas type) if its Binet formula has a similar structure to that of Fibonacci (Lucas) numbers. Such sequences are called *generalized Fibonacci polynomials* (GFPs). Remember that the Fibonacci and Lucas polynomials are defined by the following recurrence relations:

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{for } n \geq 2,$$
$$L_0(x) = 2, \quad L_1(x) = x, \quad L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad \text{for } n \geq 2.$$

Evaluating these polynomials at $x = 1$ gives the well-known Fibonacci and Lucas numbers, respectively. Other familiar examples of GFPs include Pell, Pell-Lucas, Fermat, Fermat-Lucas, both types of Chebyshev polynomials, Jacobsthal, Jacobsthal-Lucas, and both types of Morgan-Voyce polynomials (see Table 1).

The classic Catalan identities, given in [8], are expressed as

$$F_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} 5^i \quad \text{and} \quad L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2i} 5^i. \tag{1.1}$$

In this paper, we generalize the Catalan identities to include GFPs. We provide three different proofs by using combinatorial identities, generating functions, and the combinatorial algorithm of Zeilberger. In particular, we use the algorithms implemented in the software *Mathematica* by the Research Institute for Symbolic Computation (RISC).

Here, we highlight the application of computer algebra in studying such problems. Additionally, we study the sequences defined by the combinatorial sum $\sum_{i\geq 0}\binom{n}{\ell i+s}t^i$. Note that for $\ell = 2$, $t = 5$, and $s = 0, 1$, we recover the sums in the Catalan identities. We prove that the sequence defined by this combinatorial sum satisfies a recurrence relation of order ℓ, and we provide their generating functions.

Polynomial	Initial value $G_0(x) = p_0(x)$	Initial value $G_1(x) = p_1(x)$	Recursive Formula $G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x)$
Fibonacci	0	1	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Lucas	2	x	$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$
Pell	0	1	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$
Pell-Lucas-prime	1	x	$Q'_n(x) = 2xQ'_{n-1}(x) + Q'_{n-2}(x)$
Fermat	0	1	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev second kind	0	1	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Jacobsthal	0	1	$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x)$
Jacobsthal-Lucas	2	1	$j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x)$
Morgan-Voyce	0	1	$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$
Vieta	0	1	$V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$
Vieta-Lucas	2	x	$v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$

Table 1. Recurrence relation of some GFPs.

Finally, we extend Stewart's integral representation of the Fibonacci numbers [18] to polynomials by leveraging the Catalan identities. This extension further enriches the understanding and applications of these polynomial sequences.

By evaluating these polynomial sequences at $x = 1$, we can derive Catalan identities for numerical sequences (see Table 2, for examples). For instance, substituting the appropriate values into the Catalan identities provided here yields the classical Catalan identities for Fibonacci and Lucas numbers.

Sequence	Sequence	OEIS	Catalan identity
Fibonacci, $F_n(1)$	$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$	A000045	$(1/2^{n-1}) \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 5^i$
Lucas, $L_n(1)$	$2, 1, 3, 4, 7, 11, 18, 29, \dots$	A000032	$(1/2^{n-1}) \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 5^i$
Pell, $P_n(1)$	$0, 1, 2, 5, 12, 29, 70, 169 \dots$	A000129	$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 2^i$
Pell-Lucas-prime, $Q'_n(1)$	$1, 1, 3, 7, 17, 41, 99, 239 \dots$	A001333	$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 2^i$
Fermat,$\Phi_n(1)$	$0, 1, 3, 7, 15, 31, 63, 127 \dots$	A000225	$(1/2^{n-1}) \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 3^{n-2i-1}$
Fermat-Lucas, $\vartheta_n(1)$	$2, 3, 5, 9, 17, 33, 65, 129 \dots$	A000051	$(1/2^{n-1}) \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 3^{n-2i}$

Table 2. Catalan identities for some numerical sequences.

Most of the topics covered in this paper are well-suited for undergraduate students. The generalization of the Catalan identities to include generalized Fibonacci polynomials and the proofs using combinatorial identities can be accessible and engaging for undergraduate students.

These topics provide an opportunity for students to explore various techniques and approaches in combinatorics, algebra, and computer algebra systems. Additionally, the application of these concepts to polynomial sequences and the extension of integral representations offer a deeper understanding of the subject matter.

Overall, the paper presents a range of topics that can be incorporated into undergraduate course work or research projects, allowing students to delve into the world of combinatorial identities and polynomial sequences.

2. Background: The Generalized Fibonacci Polynomials

We now summarize some concepts from [4, 5, 6]. Consider two fixed nonzero polynomials $d(x)$ and $g(x)$ in $\mathbb{Q}[x]$. For $n \geq 2$, we define a second-order polynomial recurrence relation of *Fibonacci-type* as

$$\mathcal{F}_0(x) = 0, \quad \mathcal{F}_1(x) = 1, \quad \text{and} \quad \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x) \quad \text{for } n \geq 2. \tag{2.1}$$

Similarly, a second-order polynomial recurrence relation of *Lucas-type* satisfies the relation

$$\mathcal{L}_0(x) = p_0, \quad \mathcal{L}_1(x) = p_1(x), \quad \text{and} \quad \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x) \quad \text{for } n \geq 2, \tag{2.2}$$

where $|p_0| = 1$ or 2 and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$.

If $n \geq 0$ and $d^2(x) + 4g(x) \neq 0$, then the Binet formulas for the recurrence relations in (2.1) and (2.2) are given by

$$\mathcal{F}_n(x) = \frac{a^n(x) - b^n(x)}{a(x) - b(x)} \quad \text{and} \quad \mathcal{L}_n(x) = \frac{a^n(x) + b^n(x)}{\alpha}, \tag{2.3}$$

respectively. Here, we have

$$a(x) = \frac{d(x) + \sqrt{d^2(x) + 4g(x)}}{2} \quad \text{and} \quad b(x) = \frac{d(x) - \sqrt{d^2(x) + 4g(x)}}{2}. \tag{2.4}$$

Therefore,

$$a(x) + b(x) = d(x), \quad a(x)b(x) = -g(x), \quad \text{and} \quad a(x) - b(x) = \sqrt{d^2(x) + 4g(x)}. \tag{2.5}$$

(For details on the construction of the two Binet formulas, see [4].) Table 1 presents some examples of polynomial sequences of these types.

3. Catalan Identity for Generalized Fibonacci Polynomials

In this section, we aim to prove one of the main objectives of this paper, that both Catalan identities hold for GFPs of Fibonacci and Lucas type. Theorem 3.3 provides the required generalization. By evaluating $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ at, for instance, $x = 1$, we can naturally apply this identity to establish the Catalan identity for Fibonacci numbers, Lucas numbers, Pell numbers, Mersenne numbers, Fermat, and other sequences. However, by evaluating them at other values, we can derive some identities from the Online Encyclopedia of Integer Sequences OEIS [15] (see for example, Table 2).

For brevity in the rest of the paper, we present polynomials without the variable "x".

Lemma 3.1 ([6]). *Let $\mathcal{F}_n$ and $\mathcal{L}_n$ be GFPs of Fibonacci type and Lucas type, respectively. Then, for $n \geq 1$,*

(1) $$\mathcal{F}_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} d^{n-2i-1} g^i.$$

(2) $$\mathcal{L}_n = \frac{1}{\alpha} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} d^{n-2i} g^i.$$

Lemma 3.2. *Let m, n, and k be positive integers. Then, we have the following identities.*

(1) *If $0 \le m < n$, then*

$$\binom{n-m-1}{m} = 2^{2m-n+1} \sum_{j=m}^{\lfloor (n-1)/2 \rfloor} \binom{j}{m}\binom{n}{2j+1}.$$

(2) *If $0 \le k < \lfloor (n-1)/2 \rfloor$, then*

$$\binom{n}{2k+1} = \sum_{i=k}^{\lfloor (n-1)/2 \rfloor} (-1)^{i-k} 2^{n-2i-1} \binom{n-i-1}{i}\binom{i}{k}.$$

(3) *If $0 \le k < \lfloor n/2 \rfloor$, then*

$$\binom{n}{2k} = n \sum_{i=k}^{\lfloor n/2 \rfloor} \frac{(-1)^{i-k}}{(n-i)} 2^{n-2i-1} \binom{n-i}{i}\binom{i}{k}.$$

Proof. We begin by noting that Part 1 of this lemma has already been proved by Webb and Parberry [19]. We can give a computational proof by means of the Wilf-Zeilberger algorithm [12]. Let $F(n, i)$ be the expression

$$F(n,i) := \frac{2^{2m-n+1}\binom{j}{m}\binom{n}{2j+1}}{\binom{n-m-1}{m}}.$$

By the Wilf-Zeilberger algorithm, we have that $F(n, i)$ satisfies the relation

$$F(n+1,i) - F(n,i) = G(n,i+1) - G(n,i),$$

with the certificate

$$R(n,i) = \frac{(1+2j)(-j+m)}{(-2j+n)(-m+n)}.$$

That is, $R(n, i) = F(n, i)/G(n, i)$ is a rational function in both variables. Notice that if $f(n) := \sum_{i\ge 0} F(n, i)$, then $f(n + 1) - f(n) = 0$; that is, the sum is constant, in particular $f(n) = 1$. This last equality is equivalent to the desired identity.

Now, we provide the proofs for Parts 2 and 3.

Proof of Part 2. Using equation (2.4), with $x = 1$ and the first and second lines of Table 1, we have that $\alpha_1 := a(1) = (1+\sqrt{5})/2$ and $\beta_1 := b(1) = (1-\sqrt{5})/2$. Using the Catalan identity given in (1.1) and Lemma 3.1 Part 1 with $x = 1$, we obtain the identity

$$\frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (\alpha_1 - \beta_1)^{2k} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \frac{1}{4^i} [(\alpha_1-\beta_1)^2 - 1]^i.$$

We can now simplify the expression to become

$$\begin{aligned}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \frac{1}{4^i} [(\alpha_1-\beta_1)^2 - 1]^i &= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \sum_{k=0}^{i} \binom{i}{k} \frac{(-1)^k}{4^i} (\alpha_1-\beta_1)^{2k} \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=k}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{i-k}}{4^i} \binom{n-i-1}{i}\binom{i}{k} (\alpha_1-\beta_1)^{2k}.\end{aligned}$$

The proof follows by comparing coefficients of $(\alpha_1 - \beta_1)$.

Proof of Part 3. Using the Catalan identity given in (1.1) and Lemma 3.1 Part 2 with $x = 1$, we obtain the identity

$$\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}(\alpha_1-\beta_1)^{2k}=\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n}{4^i(n-i)}\binom{n-i}{i}((\alpha_1-\beta_1)^2-1)^i.$$

The remaining part of the proof follows identically to the proof of Part 2. □

The reader interested in knowing more applications of the computer algebra to prove combinatorial sums can read the references [1, 7, 11, 12, 13, 16].

The result presented in Theorem 3.3 offers an extension of the well-known Catalan identities (1.1). Catalan identities have been the subject of numerous generalizations, often relying on the utilization of the Binet formula. Although an existing proof available in [10] can be manipulated to align with our proposition and incorporate the Binet formula, we aim to provide three alternative proofs for our proposition in this paper. Additionally, a combinatorial proof for this proposition can be obtained by adapting the proof given by Rouse in [14, Theorem 2.10], which can also be found in [2].

Theorem 3.3 (Catalan identities for GFPs)**.** *For $n \geq 0$, we have the following identities.*

(1)
$$\mathcal{F}_n=\frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n}{2i+1}d^{n-2i-1}\left(4g+d^2\right)^i.$$

(2)
$$\mathcal{L}_n=\frac{1}{\alpha 2^{n-1}}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2i}d^{n-2i}\left(4g+d^2\right)^i.$$

First proof. We first prove Part 1. Using equation (2.5), we can express g in terms of a, b, and d. Then, Lemma 3.1 Part 1 implies that

$$\mathcal{F}_n=\sum_{i=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-i-1}{i}\frac{1}{4^i}[(a-b)^2-d^2]^i d^{n-2i-1}.$$

Using the binomial theorem and simplifying, we obtain

$$\mathcal{F}_n=\sum_{i=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-i-1}{i}\frac{1}{4^i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(a-b)^{2k}d^{n-2k-1}.$$

It can be verified that the right side is equal to

$$\sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor}\left[\sum_{i=k}^{\lfloor \frac{n-1}{2}\rfloor}\frac{(-1)^{i-k}}{4^i}\binom{n-i-1}{i}\binom{i}{k}\right](a-b)^{2k}d^{n-2k-1}. \tag{3.1}$$

The expression within the brackets is Lemma 3.2 Part 2, and $(a-b)^2$ corresponds to d^2+4g. Substituting these expressions into (3.1) and simplifying, we obtain the desired result.

The proof of Part 2 is similar, except we use Lemma 3.1 Part 2 and Lemma 3.2 Part 3. □

Second proof. We now present our second alternative proof using generating functions. First, we focus on proving the identity

$$2^{n-1}\mathcal{F}_n=\sum_{i=0}^{n-1}\binom{n-1}{i}d^{n-2\lfloor i/2\rfloor-1}(4g+d^2)^{\lfloor i/2\rfloor}.$$

Note that the sequence $h_n = 2^{n-1}\mathcal{F}_n$ satisfies the recurrence relation $h_n = 2dh_{n-1} + 4gh_{n-2}$ for $n \geq 2$, with initial values $h_0 = 0$ and $h_1 = 1$. Using standard techniques, we can derive the following expression for the generating function of the sequence h_n.

$$H(z) := \sum_{n\geq 0} h_n z^n = \frac{z}{1-2dz-4gz^2}.$$

On the other hand, we have

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n-1}{i}d^{n-2\lfloor i/2\rfloor-1}(4g+d^2)^{\lfloor i/2\rfloor}z^n &= \sum_{i\geq 0}(4g+d^2)^{\lfloor i/2\rfloor}\sum_{n\geq i}d^{n-2\lfloor i/2\rfloor-1}\binom{n-1}{i}z^n \\ &= \sum_{i\geq 0}(4g+d^2)^{\lfloor i/2\rfloor}d^{i-1-2\lfloor i/2\rfloor}z^i\sum_{n\geq 0}\binom{n-1+i}{n-1}(dz)^n \\ &= \sum_{i\geq 0}(4g/d^2+1)^{\lfloor i/2\rfloor}d^{i-1}z^i\frac{dz}{(1-dz)^{i+1}}.\end{aligned}$$

By considering the even and odd terms in the last sum, we obtain that

$$\sum_{i\geq 0}(4g/d^2+1)^i\frac{d^{2i}z^{2i+1}}{(1-dz)^{2i+1}} + \sum_{i\geq 0}(4g/d^2+1)^i\frac{d^{2i+1}z^{2i+2}}{(1-dz)^{2i+2}}$$

is equal to

$$\frac{z(1-dz)^2}{(1-dz)(1-2dz-4gz^2)} + \frac{dz^2}{1-2dz-4gz^2} = \frac{z}{1-2dz-4gz^2} = H(z).$$

By comparing the nth coefficient, we obtain the desired result. □

Third proof. We now present our third alternative proof using Zeilberger's creative telescoping method [12]. We denote the right side of the equality, given in the statement of the proposition, by $F(n,i)$. That is,

$$F(n,i) := \binom{n}{2i+1}d^{n-1-2i}(4g+d^2)^i.$$

By Zeilberger's algorithm, we have that $F(n,i)$ satisfies the relation

$$F(n+2,i) - 2dF(n+1,i) - 4gF(n,i) = G(n,i+1) - G(n,i), \tag{3.2}$$

with the certificate

$$R(n,i) = \frac{2d^2 i(1+2i)}{(-2i+n)(1-2i+n)}.$$

That is, $R(n,i) = F(n,i)/G(n,i)$ is a rational function in both variables. If p_n denotes the right side in the first Catalan equality, given in the statement of the proposition, then we have that summing both sides of (3.2) with respect to i yields $p_{n+2} - 2dp_{n+1} - 4gp_n = 0$. Because the sequences $h_n = 2^{n-1}\mathcal{F}_n$ and p_n satisfy the same recurrence relation and have the same initial conditions, we conclude that these sequences coincide for all positive integers n. This completes the proof. □

4. A General Combinatorial Sum

If we take the odd and even terms in the binomial sum $\sum_{i\geq 0}\binom{n}{i}5^{\lfloor i/2\rfloor}$, we obtain the sums of the Catalan identities given in (1.1), see Figure 1. This observation serves as the inspiration for our investigation in this section, where we explore this pattern in the context of a general combinatorial sum. To establish the validity of various combinatorial sums, we employ the Wilf-Zeilberger algorithm.

$$\sum_{i\geq 0}\binom{n}{i}5^{\lfloor i/2\rfloor} = 2^n F_{n+1}$$

Even terms   Odd terms

$$\sum_{i\geq 0}\binom{n}{2i}5^{i} = 2^{n-1}L_n \qquad \sum_{i\geq 0}\binom{n}{2i+1}5^{i} = 2^{n-1}F_n$$

Figure 1. Catalan identities.

For a fixed positive integer ℓ and a parameter t, we define combinatorial sequences for all $n\geq 0$ as

$$T_\ell(n;t) := \sum_{i\geq 0}\binom{n}{i}t^{\lfloor i/\ell\rfloor} \tag{4.1}$$

and

$$T_{\ell,s}(n;t) := \sum_{i\geq 0}\binom{n}{\ell i+s}t^{i}. \tag{4.2}$$

From the definitions, it is clear that $T_\ell(n;t)=\sum_{s=0}^{\ell-1}T_{\ell,s}(n;t)$. Note that $T_{2,0}(n;5)=2^{n-1}L_n$, $T_{2,1}(n;5)=2^{n-1}F_n$, and $T_2(n;5)=2^nF_{n+1}$.

The main result in this section is the generating function of the sequence $\mathcal{T}_\ell := (T_\ell(n;t))_n$. For that, we need some previous results.

Lemma 4.1. *For all* $1\leq \ell\leq n$, $\lfloor (1-s)/\ell\rfloor \leq j \leq \lfloor (n-s)/\ell\rfloor$, *and* $s\geq 0$, *we have*

$$\sum_{i\geq 0}\binom{\ell}{i}(-1)^i\binom{n-i}{\ell j+s} = \binom{n-\ell}{\ell(j-1)+s}.$$

Proof. We use the Wilf-Zeilberger algorithm [12]. Let $F(n,i)$ be the expression

$$F(n,i) := \frac{\binom{\ell}{i}(-1)^\ell\binom{n-i}{\ell j+s}}{\binom{n-\ell}{\ell(j-1)+s}}.$$

By the Wilf-Zeilberger algorithm, we have that $F(n,i)$ satisfies the relation

$$F(n+1,i)-F(n,i) = G(n,i+1)-G(n,i),$$

with the certificate

$$R(n,i) = \frac{i(1-i+n)}{(1-\ell+n)(-1+i+\ell j-n+s)}.$$

It is worth noting that if we define $f(n) := \sum_{i\geq 0} F(n,i)$, then we observe that $f(n+1) - f(n) = 0$, indicating that the sum is constant. In particular, we have $f(n) = 1$. This equality is equivalent to the desired identity. □

The combinatorial sum given in Lemma 4.1 is equivalent to

$$\binom{n}{\ell j + s} = \sum_{i\geq 1}\binom{\ell}{i}(-1)^{i-1}\binom{n-i}{\ell j+s} + \binom{n-\ell}{\ell(j-1)+s}. \tag{4.3}$$

Multiplying (4.3) by t^j and summing over all $j \geq 0$, we obtain, for all $1 \leq \ell \leq n$ and $0 \leq s \leq \ell - 1$, the identity

$$\sum_{i\geq 0}\binom{n}{\ell i+s}t^i = \sum_{i\geq 1}\binom{\ell}{i}(-1)^{i-1}\sum_{j\geq 0}\binom{n-i}{\ell j+s}t^j + t\sum_{j\geq 0}\binom{n-\ell}{\ell j+s}t^j. \tag{4.4}$$

By using the notation introduced in (4.2), identity (4.3) can be written as

$$T_{\ell,s}(n;t) = \sum_{i=1}^{\ell}\binom{\ell}{i}(-1)^{i-1}T_{\ell,s}(n-i;t) + tT_{\ell,s}(n-\ell;t). \tag{4.5}$$

It says that the sequence $\mathcal{T}_\ell = (T_\ell(n;t))_n$ satisfies a recurrence relation of order ℓ. From this recurrence relation and using the notation introduced in (4.1), we have the following theorem.

Theorem 4.2. *For all $n \geq 0$ and $\ell \geq 1$, we have*

$$\sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^i T_\ell(n+\ell-i;t) - tT_\ell(n;t) = 0.$$

The characteristic polynomial associated with the recurrence relation of the sequence $\mathcal{T}_\ell$ is given by

$$c_\ell(x) = \sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^i x^{\ell-i} - t = (x-1)^\ell - t.$$

From similar arguments, we can prove the following identity.

Lemma 4.3. *For all $n \geq 0$, we have*

$$\sum_{i=0}^{n}\binom{\ell}{i}(-1)^i\sum_{j=0}^{n-i}\binom{n-i}{j} = \sum_{i=0}^{n}(-1)^i\binom{i+\ell-n-1}{i}.$$

Theorem 4.4. *The generating function of the sequence $T_\ell(n;t)$ is the rational function*

$$T_{\ell,t}(x) := \sum_{n\geq 0} T_\ell(n;t)x^n = \frac{\sum_{k=0}^{\ell-1}\sum_{i=0}^{k}(-1)^i\binom{i+\ell-k-1}{i}x^k}{(1-x)^\ell - tx^\ell}.$$

Proof. It is evident that the generating function must be rational because the sequence $\mathcal{T}_\ell$ obeys a linear recurrence relation with constant coefficients. It is known that the denominator of the rational generating function is determined by the reflected polynomial of the characteristic polynomial (as described by Stanley in [17]). In this particular case, the polynomial is given by $x^\ell c_\ell(1/x) = x^\ell\left((\frac{1}{x}-1)^\ell - t\right) = (1-x)^\ell - tx^\ell$. Moreover, the sequence $\mathcal{T}_\ell$ is of order ℓ; the numerator of its generating function is a polynomial $p_\ell(x)$ of degree $< \ell$. This polynomial can be calculated as

$$p_\ell(x) = \left(\sum_{n=0}^{\ell-1} T_\ell(n;t)x^n\right)\left((1-x)^\ell - (-1)^\ell x^\ell\right).$$

From Lemma 4.3, we conclude that

$$p_\ell(x) = \sum_{k=0}^{\ell-1}\sum_{i=0}^{k}(-1)^i\binom{i+\ell-k-1}{i}x^k.$$

This completes the proof. □

For example, for $\ell = 1, 2, 3, 4$ we have the rational generating functions

$$T_{1,t}(x) = \frac{1}{1-(1+t)x}, \quad T_{2,t}(x) = \frac{1}{1-2x+(1-t)x^2},$$
$$T_{3,t}(x) = \frac{1-x+x^2}{1-3x+3x^2-(1+t)x^3}, \quad T_{4,t}(x) = \frac{1-2x+2x^2}{1-4x+6x^2-4x^3+(1-t)x^4}.$$

Let $p(n,m) = \sum_{i=0}^{n}(-1)^i\binom{i+n-m-1}{i}$, that is the sequence $p(n,m)$ corresponds to the coefficients of the polynomial $p_n(x)$. The array $(p(n,m))_{n,m\geq 0}$ coincides with the sequence A220074 and the first few rows are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 4 & -2 & 1 & 0 & 0 \\ 1 & -4 & 7 & -6 & 3 & 0 & 0 \\ 1 & -5 & 11 & -13 & 9 & -3 & 1 \\ 1 & -6 & 16 & -24 & 22 & -12 & 4 \end{pmatrix}.$$

This matrix was studied as the alternating Jacobsthal triangle by Lee and Oh in [9].

From a similar argument as in Theorem 4.4, we obtain Theorem 4.5.

Theorem 4.5. *The generating function of the sequence $T_{\ell,s}(n;t)$ is the rational function*

$$T_{\ell,s,t}(x) := \sum_{n\geq 0} T_{\ell,s}(n;t)x^n = \frac{\sum_{k=0}^{\ell-1}\sum_{i=0}^{k}(-1)^i\binom{\ell}{i}\sum_{r=0}^{k}\binom{k-i}{r\ell+s}t^t x^k}{(1-x)^\ell - tx^\ell}.$$

For example, consider the case of $\ell = 3$, which corresponds to the combinatorial sum $T_3(n;t) = \sum_{i\geq 0}\binom{n}{i}t^{\lfloor i/3\rfloor}$. By applying Theorems 4.4 and 4.5, we can derive the generating functions

$$(T_{3,t}(x), T_{3,0,t}(x), T_{3,1,t}(x), T_{3,2,t}(x)) = \Bigg(\frac{1-x+x^2}{1-3x+3x^2-(1+t)x^3},$$
$$\frac{1-2x+x^2}{1-3x+3x^2-x^3-tx^3}, \frac{x-x^2}{1-3x+3x^2-x^3-tx^3}, \frac{x^2}{1-3x+3x^2-x^3-tx^3}\Bigg).$$

In Table 3, we give the first few values of the sequences $T_{3,s}(n,2)$ and $T_{3,s}(n,3)$ for $s = 0, 1, 2$.

t	Sequences $(T_3(n,t), T_{3,0}(n,t), T_{3,1}(n,t), T_{3,2}(n,t))$, $n \geq 0$	OEIS
2	1, 2, 4, 9, 21, 48, 108, 243, 549, 1242, 2808	A137256
	1, 1, 1, 3, 9, 21, 45, 99, 225, 513, 1161	A052101
	0, 1, 2, 3, 6, 15, 36, 81, 180, 405, 918	A052102
	0, 0, 1, 3, 6, 12, 27, 63, 144, 324, 729	A052103
3	1, 2, 4, 9, 21, 48, 108, 243, 549, 1242, 2808	
	1, 1, 1, 4, 13, 31, 70, 169, 421, 1036, 2521	A097122
	0, 1, 2, 3, 7, 20, 51, 121, 290, 711, 1747	
	0, 0, 1, 3, 6, 13, 33, 84, 205, 495, 1206	

Table 3. Some values of $T_{3,s}(n,2)$ and $T_{3,s}(n,3)$.

5. Integral Representation of Generalized Fibonacci Polynomials

The function $A(x,y) := (d + y\sqrt{d^2+4g})/2$ can be seen as a generalization of the golden ratio $(1+\sqrt{5})/2$. In particular, when we set $y = 1$, we have $A(x,1) = (d+\sqrt{d^2+4g})/2 = a(x)$ as defined in (2.4). Therefore, when we choose $g = d = 1$, we recover the golden ratio.

In this section, we integrate $A(x,y)^{n-1}$ to obtain a representation for the GFPs. Moreover, Theorem 3.3 can be used to extend the proof of Corollary 5.1, which was presented in [18] for Fibonacci numbers, and consequently derive a second proof of Proposition 5.1.

Proposition 5.1. *If $n \geq 2$, then*

$$\mathcal{F}_n = \frac{n}{2^n}\int_{-1}^{1}\left(d + y\sqrt{d^2+4g}\right)^{n-1} dy.$$

Proof from Catalan identity. By the binomial theorem and the Catalan identity, we have

$$\begin{aligned}\frac{n}{2}\int_{-1}^{1}(d + y\sqrt{d^2+4g})^{n-1}dy &= \frac{n}{2}\int_{-1}^{1}\sum_{i=0}^{n-1}\binom{n-1}{i}d^{n-1-i}y^i\left(\sqrt{d^2+4g}\right)^i dy\\ &= \frac{n}{2^n}\sum_{i=0}^{n-1}\binom{n-1}{i}d^{n-1-i}\left(\sqrt{d^2+4g}\right)^i\int_{-1}^{1}y^i dy\\ &= \frac{n}{2}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{2i}d^{n-1-2i}\left(\sqrt{d^2+4g}\right)^{2i}\frac{2}{2i+1}\\ &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}d^{n-1-2i}(d^2+4g)^i = 2^{n-1}\mathcal{F}_n. \qquad \square\end{aligned}$$

Second proof. We can start by noting that if

$$s(x,y) := \frac{\Big(y\big(a-b\big)+d\Big)^n}{a-b},$$

then, taking the partial derivative of $s(x,y)$ with respect to y, we have

$$\frac{\partial s(x,y)}{\partial y} = n(y\sqrt{d^2+4g}+d)^{n-1}.$$

From the Fundamental Theorem of Calculus, we have that the integral in the statement of the proposition is equal to

$$\frac{\Big((1)\big(a-b\big)+d\Big)^n}{a-b}-\frac{\Big((-1)\big(a-b\big)+d\Big)^n}{a-b}.$$

Simplifying this expression using equations (2.5), we obtain equation (2.3). This completes the proof. □

Corollary 5.1 ([18])**.** *If F_n is the nth Fibonacci number, then*

$$F_n=\frac{n}{2^n}\int_{-1}^{1}(1+y\sqrt{5})^{n-1}dy.$$

It is worth noting that as a corollary of Proposition 5.1, using [5, Identity 2], it is possible to obtain an integral expression for $\mathcal{L}_n$.

It should be noted that the second proof presented in Proposition 5.1 can be adapted to derive a more general result. By changing the limits of integration, we can obtain an integral representation for a binomial expression involving Fibonacci numbers. (See other integral expressions in [3].)

Proposition 5.2. *If $k\geq 2$, then*

$$\sum_{i=0}^{n}(-1)^{n+1}\binom{n}{i}(k+1)^i(k-1)^{n-i}g^i\mathcal{F}_{n-2i}=n\int_{-k}^{k}\big(y\sqrt{d^2+4g}+d\big)^{n-1}dy.$$

Corollary 5.2. *If F_n is the nth Fibonacci number and $k\geq 2$, then*

$$\sum_{i=0}^{n}(-1)^{n+1}\binom{n}{i}(k+1)^i(k-1)^{n-i}F_{n-2i}=n\int_{-k}^{k}\Big(\sqrt{5}y+1\Big)^{n-1}\,dy.$$

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