CLOSING AN OPEN PROBLEM ON NEGATIVE BASE HAPPY NUMBERS

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ABSTRACT. For $b \le -2$, let $S_{2,b} : \mathbb{Z} \to \mathbb{Z}_{\ge 0}$ be the function taking an integer to the sum of the squares of the digits of its base b expansion. An integer a is a b-happy number if there exists $k \in \mathbb{Z}^+$ such that $S_{2,b}^k(a) = 1$. It has been shown that for $b \le -5$ and odd, there exist arbitrarily long finite arithmetic sequences with constant difference 2 of b-happy numbers and that for $b \in \{-4, -6, -8, -10\}$, there exist arbitrarily long finite sequences of consecutive b-happy numbers. In this work, we complete this result, proving that, as conjectured, for all even $b \le -4$, there exist arbitrarily long finite sequences of consecutive b-happy numbers.

1. Introduction

In 1994, Richard Guy [5] asked for the maximal length of sequences of consecutive happy numbers, if such a maximum exists. In 2000, El-Sedy and Siksek [1] proved that there exist arbitrarily long finite sequences of happy numbers. In 2007, Grundman and Teeple [4] extended this result to b-happy numbers for all bases $b \ge 2$, with the observation that for odd bases, only 2-consecutive sequences (arithmetic sequences with constant difference 2) are possible. In 2008, Pan [6] proved the existence of arbitrarily long finite seequences of e-power b-happy number for all bases $b \ge 2$ and exponents $e \ge 2$ for which any consecutive e-power b-happy numbers exist. And in 2009, Zhou and Cai [7] generalized Pan's work to the remaining cases, proving the existence of d-consecutive sequences of arbitrary length, where d is the best possible. Negative base happy numbers with exponent 2 were first considered in 2018 by Grundman and Harris [2], who proved the following theorem and conjectured that the third part should generalize to all bases $b \le -4$.

Theorem 1.1 (Grundman & Harris)**.** *Let $b \le -2$.*

1. *There exist infinitely long sequences of 3-consecutive −2-happy numbers. In particular, $a \in \mathbb{Z}$ is −2-happy if and only if $a \equiv 1 \pmod 3$.*
2. *There exist infinitely long sequences of 2-consecutive −3-happy numbers. In particular, $a \in \mathbb{Z}$ is −3-happy if and only if $a \equiv 1 \pmod 2$.*
3. *For $b \in \{-4, -6, -8, -10\}$, there exist arbitrarily long finite sequences of consecutive b-happy numbers.*
4. *For b odd, there exist arbitrarily long finite sequences of 2-consecutive b-happy numbers.*

In this work, we show that the conjecture is correct, proving the following new theorem.

Theorem 1.2. *For $b \le -4$ and even, there exist arbitrarily long finite sequences of consecutive b-happy numbers.*

In the following section, we present definitions and initial lemmas. In Section 3, we prove two key results and then the main theorem, stated above.

2. Definitions and Preliminary Results

We begin with the definition of b-happy numbers and the corresponding functions for bases $b \leq -2$, as given in [3] and then adapted in [2]. Note that, unlike the case when b is positive, the domains of the functions include all integers.

Let $b \leq -2$. Define the function $S_{2,b} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by $S_{2,b}(0) = 0$ and for $a = \sum_{i=0}^{n} a_i b^i$ with $a_n \neq 0$ and $0 \leq a_i \leq |b| - 1$, for each $0 \leq i \leq n$,

$$S_{2,b}(a) = \sum_{i=0}^{n} a_i^2.$$

An integer a is a *b-happy number* if, for some $k \in \mathbb{Z}^+$, $S_{2,b}^k(a) = 1$.

The following definitions are from [4]. Fix $b \leq -2$. Let

$$U_{2,b} = \left\{ u \in \mathbb{Z}^+ \,\middle|\, \text{for some } m \in \mathbb{Z}^+,\ S_{2,b}^m(u) = u \right\}.$$

We say that a finite set T is *$(2,b)$-good* if, for each $u \in U_{2,b}$, there exist $n,\ k \in \mathbb{Z}^+$ such that for each $t \in T$, $S_{2,b}^k(t+n) = u$. Finally, let $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $I(t) = t + 1$.

We will use the following lemma, which is proved in more generality in [2, Lemma 8].

Lemma 2.1. *Fix $b \leq -2$. Let $T \subseteq \mathbb{Z}^+$ be finite. Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the composition of a finite sequence of the functions $S_{2,b}$ and I. If $F(T)$ is $(2,b)$-good, then T is $(2,b)$-good.*

Lemma 2.2, combined with Lemma 2.1, enables significant simplification of the proofs in the following section.

Lemma 2.2. *Fix $b \leq -7$. For each finite subset $A \subseteq \mathbb{Z}^+$, there exists $k \in \mathbb{Z}^+$ such that for each $a \in A$, $S_{2,b}^k(a) < 2b^2$.*

Proof. By [2, Theorem 1], if $a > (|b|-1)(b^2+b+1)$, then $S_{2,b}(a) < a$. So, there exists a $k_1 \in \mathbb{Z}^+$ such that for each $a \in A$, $S_{2,b}^{k_1}(a) \leq (|b|-1)(b^2+b+1)$. It follows that, for each $a \in A$, $S_{2,b}^{k_1+1}(a) \leq 3(|b|-1)^2 = 3b^2 + 6b + 3$, and so $S_{2,b}^{k_1+2}(a) \leq S_{2,b}(3b^2 + (|b|-1)b + (|b|-1)) = 9 + 2(|b|-1)^2 = 2b^2 + 4b + 11 < 2b^2$. The lemma follows. □

3. Main Results

In this section, we prove that for $b \leq -4$, any finite set of positive integers is $(2,b)$-good. This then allows us to prove Theorem 1.2, the main theorem of this paper. Key to each of these proofs is the following lemma.

Lemma 3.1. *Fix $b \leq -12$ and even. For each $0 < v < 2b^2$, there exists some $0 \leq w < v$ and $0 \leq c \leq |b| - 2$ such that*

$$S_{2,b}((|b|-1)b + v - w - 1) - S_{2,b}((|b|-1)b^2 + |b| - 1 - w) + 5 + 2c \equiv 0 \pmod{(|b|+1)}. \quad (3.1)$$

Proof. Fix $b \leq -12$ and $0 < v < 2b^2$. Note that because b is even, 2 is invertible modulo $|b|+1$, and so for any $0 \leq w < v$, there exists a unique $c = c(v,w)$ satisfying (3.1) with $0 \leq c \leq |b|$. Therefore, it suffices to show that for at least one such w, $c(v,w) \leq |b|-2$. For a contradiction, suppose to the contrary that for each $0 \leq w < v$, $c(v,w) = |b|-1$ or $|b|$.

Set $v = v_2 b^2 + v_1 b + v_0$ with $0 \leq v_i < |b|$ for each i. By assumption, $v_2 \leq 2$ and if $v_2 = 2$, then $v_1 > 0$. Noting that $|b| \equiv -1 \pmod{(|b|+1)}$, it is straightforward to verify that $c(1,0) = |b|/2 < |b| - 1$ and $c(2,1) = 2 < |b| - 1$. So, we may assume that $v \geq 3$.

If $v_0 \geq 2$, then for $0 \leq w \leq 1$,

$$S_{2,b}((|b|-1)b + v - w - 1) = S_{2,b}(v_2 b^2 + (v_1 + |b| - 1)b) + (v_0 - w - 1)^2.$$

So for $w = 0$, using (3.1),

$$0 \equiv S_{2,b}(v_2b^2 + (v_1 + |b| - 1)b) + (v_0 - 1)^2 - 2(|b| - 1)^2 + 5 + 2c(v, 0) \pmod{(|b| + 1)} \tag{3.2}$$

and for $w = 1$, using (3.1),

$$0 \equiv S_{2,b}(v_2b^2 + (v_1 + |b| - 1)b) + (v_0 - 2)^2 - (|b| - 1)^2 - (|b| - 2)^2 + 5 + 2c(v, 1) \pmod{(|b| + 1)}.$$

Subtracting and simplifying yields $v_0 \equiv c(v, 1) - c(v, 0) - 1 \pmod{(|b|+1)}$. Because $2 \leq v_0 < |b|$ and each of $c(v, 0)$ and $c(v, 1)$ is by assumption equivalent to −1 or −2 modulo $|b| + 1$, this implies that $v_0 = |b| - 1$, $c(v, 0) = |b|$, and $c(v, 1) = |b| - 1$. So by (3.2),

$$S_{2,b}(v_2b^2 + (v_1 + |b| - 1)b) \equiv -(|b| - 2)^2 + 2(|b| - 1)^2 - 5 - 2|b| \equiv -4 \pmod{(|b| + 1)}.$$

Now, because $v_0 = |b| - 1$, we may let $w = 2$ in (3.1), resulting in

$$\begin{aligned} 0 &\equiv S_{2,b}(v_2b^2 + (v_1 + |b| - 1)b) + (|b| - 4)^2 - (|b| - 1)^2 - (|b| - 3)^2 + 5 + 2c(v, 2) \\ &\equiv 2c(v, 2) + 6 \pmod{(|b| + 1)}, \end{aligned}$$

and so, $c(v, 2) = |b| - 2 < |b| - 1$, a contradiction.

Thus, $v \geq 3$ and $v_0 = 0$ or 1. Hence, $v \geq b$. Because $v < 2b^2$, $v_2 = 1$ or 2, and if $v_2 = 2$, then $v_1 > 0$.

If $v_0 = 1$, then for $1 \leq w \leq 3$,

$$\begin{aligned} S_{2,b}((|b| - 1)b + v - w - 1) &= S_{2,b}(v_2b^2 + (|b| - 1 + v_1)b + v_0 - w - 1) \\ &= S_{2,b}((v_2 - 1)b^2 + v_1b + |b| - w) \\ &= S_{2,b}((v_2 - 1)b^2 + v_1b) + (|b| - w)^2. \end{aligned}$$

Therefore, using (3.1) with each of $w = 1$, 2, and 3,

$$0 \equiv S_{2,b}((v_2 - 1)b^2 + v_1b) + (|b| - 1)^2 - (|b| - 1)^2 - (|b| - 2)^2 + 5 + 2c(v, 1) \pmod{(|b| + 1)},$$

$$0 \equiv S_{2,b}((v_2 - 1)b^2 + v_1b) + (|b| - 2)^2 - (|b| - 1)^2 - (|b| - 3)^2 + 5 + 2c(v, 2) \pmod{(|b| + 1)},$$

and

$$0 \equiv S_{2,b}((v_2 - 1)b^2 + v_1b) + (|b| - 3)^2 - (|b| - 1)^2 - (|b| - 4)^2 + 5 + 2c(v, 3) \pmod{(|b| + 1)}.$$

Subtracting and simplifying yields $c(v, 2) - c(v, 1) \equiv 1 \pmod{(|b| + 1)}$ and $c(v, 3) - c(v, 2) \equiv 1 \pmod{(|b|+1)}$. But this implies that at least one of $c(v, 1)$, $c(v, 2)$, and $c(v, 3)$ is not congruent to −1 or −2, a contradiction.

Therefore, $v_0 = 0$. Letting $w = 0$ in (3.1), we have

$$\begin{aligned} 0 &\equiv S_{2,b}((|b| - 1)b + v - 1) - S_{2,b}((|b| - 1)b^2 + |b| - 1) + 5 + 2c(v, 0) \\ &\equiv S_{2,b}((v_2 - 1)b^2 + v_1b + (|b| - 1)) - S_{2,b}((|b| - 1)b^2 + |b| - 1) + 5 + 2c(v, 0) \\ &\equiv ((v_2 - 1)^2 + v_1^2 + (-2)^2) - ((-2)^2 + (-2)^2) + 5 + 2c(v, 0) \pmod{(|b| + 1)}, \end{aligned}$$

and so,

$$(v_2 - 1)^2 + v_1^2 + 1 + 2c(v, 0) \equiv 0 \pmod{(|b| + 1)}. \tag{3.3}$$

Recalling that by assumption $v_2 = 1$ or 2 and $c(v, 0) = |b| - 1$ or $|b|$, (3.3) implies that $v_1 \neq |b| - 1$.

Now letting $w = |b| < v$, (3.1) becomes

$$\begin{aligned} 0 &\equiv S_{2,b}((|b| - 1)b + v - |b| - 1) - S_{2,b}((|b| - 1)b^2 + |b| - 1 - |b|) + 5 + 2c(v, |b|) \\ &\equiv S_{2,b}((v_2 - 1)b^2 + (v_1 + 1)b + (|b| - 1)) - S_{2,b}((|b| - 1)b^2 + b + (|b| - 1)) + 5 + 2c(v, |b|) \\ &\equiv ((v_2 - 1)^2 + (v_1 + 1)^2 + (-2)^2) - ((-2)^2 + 1 + (-2)^2) + 5 + 2c(v, |b|) \pmod{(|b| + 1)}, \end{aligned}$$

and so, $(v_2-1)^2+(v_1+1)^2+2c(v,|b|) \equiv 0 \pmod{(|b|+1)}$. Subtracting (3.3) and simplifying yields $v_1+c(v,|b|)-c(v,0) \equiv 0 \pmod{(|b|+1)}$. Hence, $v_1 = 0$ or 1. Again using (3.3) and recalling that $v < 2b^2$, we find that $v = b^2+b$.

Finally, verifying that $c(b^2+b, 2|b|) = |b|-2 < |b|-1$ completes the proof. □

We now use the fact that $0 \le c(v,w) < |b|-1$ to prove that every finite set of positive integers is $(2,b)$-good.

Theorem 3.2. *If $b \le -4$ is even, then every finite set of positive integers is $(2,b)$-good.*

Proof. By [2, Theorem 12], the theorem holds for even $-10 \le b \le -4$. So we fix even $b \le -12$.

Fix a finite set of positive integers T. If T is empty, then vacuously it is $(2,b)$-good. If $T = \{t\}$, then given $u \in U_{2,b}$, by definition, there exist $x \in \mathbb{Z}^+$ such that $S_{2,b}(x) = u$. Fix some $r \in \mathbb{Z}^+$ such that $t \le b^{2r}x$. Then, letting $n = b^{2r}x - t$ and $k = 1$, because $S^k_{2,b}(t+n) = S_{2,b}(b^{2r}x)) = u$, T is $(2,b)$-good by definition.

Now assume that $|T| > 1$ and assume, by induction, that any set of fewer than N positive integers is $(2,b)$-good. Applying Lemmas 2.1 and 2.2, we may assume that for each $t \in T$, $t < 2b^2$. Fix $t_1 > t_2 \in T$.

Let $v = t_1 - t_2 < 2b^2$. By Lemma 3.1, there exists some $0 \le w \le v$ and $0 \le c < |b|-1$ such that (3.1) holds. Let

$$m = cb^4 + (|b|-1)b^2 + (|b|-1) - t_2 - w \ge 0.$$

Then, since $|b| = -b$,

$$\begin{aligned} S_{2,b}\left(t_1+m\right) &= S_{2,b}\left(cb^4+(|b|-1)b^2+(|b|-1)+v-w\right) \\ &= S_{2,b}\left((c+1)b^4+(|b|-1)b^3+(|b|-1)b+v-w-1\right) \\ &= (c+1)^2+(|b|-1)^2+S_{2,b}\left((|b|-1)b+v-w-1\right). \end{aligned}$$

By the choice of c, we have

$$\begin{aligned} S_{2,b}\left(t_1+m\right) &\equiv (c+1)^2+(|b|-1)^2+S_{2,b}((|b|-1)b^2+|b|-1-w)-5-2c \\ &\equiv c^2+S_{2,b}\left((|b|-1)b^2+(|b|-1)-w\right) \\ &\equiv S_{2,b}\left(cb^4+(|b|-1)b^2+(|b|-1)-w\right) \\ &\equiv S_{2,b}(t_2+m) \pmod{(|b|+1}. \end{aligned}$$

Now, if $S_{2,b}\left(t_1+m\right) = S_{2,b}(t_2+m)$, then $|S_{2,b}I^m(T)| < |T|$ and so, $S_{2,b}I^m(T)$ is $(2,b)$-good, implying, by Lemma 2.1, that T is $(2,b)$-good, as desired.

If not, then fix $t_3 > t_4 \in S_{2,b}I^m(T)$ with

$$\{t_3,t_4\} = \{S_{2,b}\left(t_1+m\right), S_{2,b}(t_2+m)\}.$$

Fix $v' \in \mathbb{Z}^+$ such that $t_4 - t_3 = v'(b-1)$. Choose $r \in \mathbb{Z}^+$ such that $b^{2(r-1)} > |b|v' + t_3$. Let $m' = b^{2r} + v' - t_3 > 0$. Then

$$I^{m'}(t_3) = t_3 + b^{2r} + v' - t_3 = b^{2r} + v'$$

and

$$I^{m'}(t_4) = t_4 + b^{2r} + v' - t_3 = b^{2r} + v' + v'(b-1) = b^{2r} + bv'.$$

Because $b^{2(r-1)} > |b|v'$, it follows that $I^m(t_1)$ and $I^m(t_2)$ have the same nonzero digits. Thus, letting $F' = S_{2,b}I^{m'}S_{2,b}I^m$, $F'(t_1) = F'(t_2)$. Because $|F'(T)| < |T|$, $F'(T)$ is $(2,b)$-good and, by Lemma 2.1, T is $(2,b)$-good, completing the proof. □

The main theorem now follows.

Proof of Theorem 1.2. Fix $b < -4$ and even. Given $N \in \mathbb{Z}^+$, let $T = \{1, 2, 3, \ldots, N\}$. By Theorem 3.2, T is $(2, b)$-good. By the definition of $(2, b)$-good, there exist n, $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S^k_{2,b}(t + n) = 1$. Thus, $T + n = \{1 + n, 2 + n, \ldots, N + n\}$ is a sequence of N consecutive b-happy numbers, as desired. □

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