SUMS INVOLVING A FAMILY OF JACOBSTHAL POLYNOMIAL SQUARES

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Abstract. We explore the Jacobsthal version of an infinite sum involving gibonacci polynomial squares and its implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, $D = \sqrt{4x+1}$, and $2w = 1 + D$. Then $\alpha(1/\sqrt{x}) = \dfrac{1+D}{2\sqrt{x}} = \dfrac{w}{\sqrt{x}}$.

2. Gibonacci Polynomial Sum

Before presenting an interesting gibonacci sum, again in the interest of brevity and expediency, we now let [5, 6]

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \nu^* = \begin{cases} 1, & \text{if } g_n = f_n; \\ -1, & \text{otherwise;} \end{cases} \quad \text{and} \quad D^* = \begin{cases} 1, & \text{if } c_n = J_n; \\ D^2, & \text{otherwise.} \end{cases}$$

Using these tools as building blocks, we established the following result in [5], the cornerstone of our discourse.

Theorem 2.1. *Let k, p, r, and t be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu^* f_r f_{2pk}}{g^2_{(2pn+t-p)k} - (-1)^{tk}\mu\nu^* f^2_{pk}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{2.1}$$

The goal of our discourse is to explore the Jacobsthal counterpart of this sum.

3. Jacobsthal Polynomial Sum

To realize our objective, we will employ the gibonacci-Jacobsthal relationships in Section 1. To this end, in the interest of brevity and clarity, we let A denote the fractional expression on the left side of the given gibonacci equation and B that on its right side, and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal equation, as in [5, 6].

With this short background, we now begin our endeavor.

Proof. *Case 1.* Suppose $g_n = f_n$. We have $A = \dfrac{(-1)^{tk} f_r f_{2pk}}{f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{(2pn+t)k-2+r/2}$. We get

$$\begin{aligned}
A &= \frac{(-1)^{tk} x^{(2pn+t-p)k-1}\left[x^{(r-1)/2} f_r\right]\left[x^{(2pk-1)/2} f_{2pk}\right]}{x^{pk-1+r/2}\left\{x^{[(2pn+t-p)k-1]/2} f_{(2pn+t-p)k}\right\}^2 - (-1)^{tk} x^{(2pn+t-p)k-1+r/2}\left[x^{(pk-1)/2} f_{pk}\right]^2}\\
&= \frac{(-1)^{tk} x^{(2pn+t-2p)k-r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k} - (-1)^{tk} x^{(2pn+t-2p)k} J^2_{pk}};\\
\text{LHS} &= \sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(2pn+t-2p)k-r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k} - (-1)^{tk} x^{(2pn+t-2p)k} J^2_{pk}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

We now turn to $B = \dfrac{f_{tk+r}}{f_{tk}} - \alpha^r$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{(tk+r-1)/2}$ yields

$$\begin{aligned}
B &= \frac{x^{(tk+r-1)/2} f_{tk+r}}{x^{r/2}\left[x^{(tk-1)/2} f_{tk}\right]} - \frac{w^r}{x^{r/2}};\\
\text{RHS} &= \frac{J_{tk+r}}{x^{r/2} J_{tk}} - \frac{w^r}{x^{r/2}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

By equating the two sides, we get the Jacobsthal version of equation (2.1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(2pn+t-2p)k} J_r J_{2pk}}{J^2_{(2pn+t-p)k} - (-1)^{tk} x^{(2pn+t-2p)k} J^2_{pk}} = \frac{J_{tk+r}}{J_{tk}} - w^r, \tag{3.1}$$

where $c_n = c_n(x)$.

Next, we pursue the Jacobsthal-Lucas version of Theorem 2.1.

Case 2. With $g_n = l_n$, we have $A = \dfrac{(-1)^{tk+1}\Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}}$. Again, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{(2pn+t-p)k}$, we have

$$\begin{aligned}
A &= \frac{(-1)^{tk+1}\frac{D^2}{x}\cdot x^{[(2pn+t-2p)k+1-r/2]}\left[x^{(r-1)/2} f_r\right]\left[x^{(2pk-1)/2} f_{2pk}\right]}{\left\{x^{[(2pn+t-p)k]/2} l_{(2pn+t-p)k}\right\}^2 + (-1)^{tk}\frac{D^2}{x} x^{(2pn+t-2p)k+1}\left[x^{(pk-1)/2} f_{pk}\right]^2}\\
&= \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k-r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}};\\
\text{LHS} &= \sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k-r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This time, we have $B = \dfrac{l_{tk+r}}{l_{tk}} - \alpha^r$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{(tk+r)/2}$. This yields

$$\begin{aligned} B &= \frac{x^{(tk+r)/2} l_{tk+r}}{x^{r/2}[x^{tk/2} l_{tk}]} - \frac{w^r}{x^{r/2}}; \\ \text{RHS} &= \frac{j_{tk+r}}{x^{r/2} j_{tk}} - \frac{w^r}{x^{r/2}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the corresponding Jacobsthal-Lucas version:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k} J_r J_{2pk}}{j^2_{(2pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}} = \frac{j_{tk+r}}{j_{tk}} - w^r, \tag{3.2}$$

where $c_n = c_n(x)$. □

Using equations (3.1) and (3.2), we get the Jacobsthal version of Theorem 2.1, as the following theorem features.

Theorem 3.1. *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k} J^2_{pk}} = \frac{c_{tk+r}}{c_{tk}} - w^r. \tag{3.3}$$

By employing the gibonacci-Jacobsthal relationships in a compact way, we showcase an alternate proof of this theorem.

3.1. **A Sophisticated Method.** To begin with, we let

$$d = \frac{1+\nu^*}{4} = \begin{cases} 1/2, & \text{if } g_n = f_n; \\ 0, & \text{otherwise.} \end{cases}$$

It follows, from the gibonacci-Jacobsthal links, that

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}}; \qquad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{n/2}}; \qquad g_n(1/\sqrt{x}) = \frac{c_n(x)}{x^{n/2-d}}.$$

With these new tools at our disposal, we are ready for the alternate proof.

Proof. Replacing x with $1/\sqrt{x}$ in the rational expression on the left side of equation (2.1) and using the above substitutions, we get

$$\begin{aligned} A &= \frac{(-1)^{tk} \mu\nu^* J_r / x^{(r-1)/2} \cdot J_{2pk}/x^{(2pk-1)/2}}{\left[c_{(2pn+t-p)k}/x^{\frac{(2pn+t-p)k}{2}-d}\right]^2 - (-1)^{tk}\mu\nu^* \left[\frac{J_{pk}}{x^{(pk-1)/2}}\right]^2} \\ &= \frac{(-1)^{tk}\mu\nu^* J_r J_{2pk} \cdot x^{(2pn+t-p)k-2d-\frac{2pk+r-2}{2}}}{c^2_{(2pn+t-p)k} - (-1)^{tk}\mu\nu^* J^2_{pk} \cdot x^{[(2pn+t-p)k-2d-(pk-1)}} \\ &= \frac{(-1)^{tk}\mu\nu^* J_r J_{2pk} \cdot x^{(2pn+t-2p)k-\frac{\nu^*+r-1}{2}}}{c^2_{(2pn+t-p)k} - (-1)^{tk}\mu\nu^* J^2_{pk} \cdot x^{(2pn+t-2p)k+\frac{1-\nu^*}{2}}}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k-r/2} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k} J^2_{pk}}, \end{aligned}$$

where $c_n = c_n(x)$.

The right side of equation (2.1) yields

$$\begin{aligned} B &= \frac{g_{tk+r}}{g_{tk}} - \alpha^r \\ &= \frac{c_{tk+r}/x^{\frac{tk+r}{2}-d}}{c_{tk}/x^{\frac{tk}{2}-d}} - \frac{w^r}{x^{r/2}}; \\ \text{RHS} &= \frac{c_{tk+r}}{x^{r/2}c_{tk}} - \frac{w^r}{x^{r/2}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides yields the same Jacobsthal version, as expected:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^* \nu^* x^{(2pn+t-2p)k} J^2_{pk}} = \frac{c_{tk+r}}{c_{tk}} - w^r,$$

where $c_n = c_n(x)$. □

We now explore a host of gibonacci and Jacobsthal implications of Theorem 3.1.

3.2. **Gibonacci and Jacobsthal Implications.** With $J_n(1) = F_n$ and $j_n(1) = L_n$, Theorem 3.1 yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} F_r F_{2pk}}{F^2_{(2pn+t-p)k} - (-1)^{tk} F^2_{pk}} = \frac{F_{tk+r}}{F_{tk}} - \alpha^r; \tag{3.4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 5 F_r F_{2pk}}{L^2_{(2pn+t-p)k} + (-1)^{tk} 5 F^2_{pk}} = \frac{L_{tk+r}}{L_{tk}} - \alpha^r; \tag{3.5}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} 2^{(2pn+t-2p)k} J_r J_{2pk}}{J^2_{(2pn+t-p)k} - (-1)^{tk} 2^{(2pn+t-2p)k} J^2_{pk}} = \frac{J_{tk+r}}{J_{tk}} - 2^r; \tag{3.6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 9 \cdot 2^{(2pn+t-2p)k} J_r J_{2pk}}{j^2_{(2pn+t-p)k} + (-1)^{tk} 9 \cdot 2^{(2pn+t-2p)k} J^2_{pk}} = \frac{j_{tk+r}}{j_{tk}} - 2^r. \tag{3.7}$$

Let $p = 3$, $r = 1$, and $t \leq 6$. With $k = 1$, equations (3.4) and (3.5) yield

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{F^2_{6n-2}+4} &= -\frac{1}{16} + \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n-2}-20} &= \frac{1}{16} - \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{1}{F^2_{6n-1}-4} &= \frac{3}{16} - \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n-1}+20} &= -\frac{1}{48} + \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{1}{F^2_{6n}+4} &= -\frac{1}{8} + \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n}-20} &= \frac{1}{32} - \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{1}{F^2_{6n+1}-4} &= \frac{7}{48} - \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n+1}+20} &= -\frac{3}{112} + \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{1}{F^2_{6n+2}+4} &= -\frac{11}{80} + \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n+2}-20} &= \frac{5}{176} - \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{1}{F^2_{6n+3}-4} &= \frac{9}{64} - \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L^2_{6n+3}+20} &= -\frac{1}{36} + \frac{\sqrt{5}}{80}. \end{aligned}$$

With $k = 2$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n-2)}^2 - 64} = \frac{1}{96} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-2)}^2 + 320} = -\frac{1}{864} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n-1)}^2 - 64} = \frac{7}{864} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)}^2 + 320} = -\frac{1}{672} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n)}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n)}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n+1)}^2 - 64} = \frac{47}{6,048} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+1)}^2 + 320} = -\frac{7}{4,512} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n+2)}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+2)}^2 + 320} = -\frac{55}{35,424} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n+3)}^2 - 64} = \frac{161}{20,736} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)}^2 + 320} = -\frac{1}{644} + \frac{\sqrt{5}}{1,440}.$$

Next, we showcase the Jacobsthal counterparts of these gibonacci sums.
With $k = 1$, equations (3.6) and (3.7) yield

$$\sum_{n=1}^{\infty} \frac{2^{6n-5}}{J_{6n-2}^2 + 9 \cdot 2^{6n-5}} = \frac{1}{21}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-5}}{j_{6n-2}^2 - 81 \cdot 2^{6n-5}} = \frac{1}{63};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-4}}{J_{6n-1}^2 - 9 \cdot 2^{6n-4}} = \frac{1}{21}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-4}}{j_{6n-1}^2 + 81 \cdot 2^{6n-4}} = \frac{1}{315};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-3}}{J_{6n}^2 + 9 \cdot 2^{6n-3}} = \frac{1}{63}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-3}}{j_{6n}^2 - 81 \cdot 2^{6n-3}} = \frac{1}{441};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-2}}{J_{6n+1}^2 - 9 \cdot 2^{6n-2}} = \frac{1}{105}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-2}}{j_{6n+1}^2 + 81 \cdot 2^{6n-2}} = \frac{1}{1,071};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-1}}{J_{6n+2}^2 + 9 \cdot 2^{6n-1}} = \frac{1}{231}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-1}}{j_{6n+2}^2 - 81 \cdot 2^{6n-1}} = \frac{1}{1,953};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n}}{J_{6n+3}^2 - 9 \cdot 2^{6n}} = \frac{1}{441}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n}}{j_{6n+3}^2 + 81 \cdot 2^{6n}} = \frac{1}{4,095}.$$

Using $k = 2$, we get

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{J_{2(6n-2)}^2 - 441 \cdot 2^{6n-5}} = \frac{1}{1,365}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{j_{2(6n-2)}^2 + 3,969 \cdot 2^{6n-5}} = \frac{1}{20,475};$$
$$\sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{J_{2(6n-1)}^2 - 441 \cdot 2^{6n-4}} = \frac{1}{6,825}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{j_{2(6n-1)}^2 + 3,969 \cdot 2^{6n-4}} = \frac{1}{69,615};$$
$$\sum_{n=1}^{\infty} \frac{2^{2(6n-3)}}{J_{2(6n)}^2 - 441 \cdot 2^{6n-3}} = \frac{1}{28,665}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-3)}}{j_{2(6n)}^2 + 3,969 \cdot 2^{6n-3}} = \frac{1}{266,175};$$
$$\sum_{n=1}^{\infty} \frac{2^{2(6n-2)}}{J_{2(6n+1)}^2 - 441 \cdot 2^{6n-2}} = \frac{1}{116,025}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-2)}}{j_{2(6n+1)}^2 + 3,969 \cdot 2^{6n-2}} = \frac{1}{1,052,415};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-1)}}{J^2_{2(6n+2)} - 441 \cdot 2^{6n-1}} = \frac{1}{465,465}; \quad \sum_{n=1}^{\infty} \frac{2^{2(6n-1)}}{j^2_{2(6n+2)} + 3,969 \cdot 2^{6n-1}} = \frac{1}{4,197,375};$$
$$\sum_{n=1}^{\infty} \frac{2^{2(6n)}}{J^2_{2(6n+3)} - 441 \cdot 2^{6n}} = \frac{1}{1,863,225}; \quad \sum_{n=1}^{\infty} \frac{2^{2(6n)}}{j^2_{2(6n+3)} + 3,969 \cdot 2^{6n}} = \frac{1}{16,777,215}.$$

3.3. **Gibonacci Delights.** Using the above gibonacci sums, we can extract dividends.

$$\sum_{n=2}^{\infty} \frac{1}{F^2_{2n}+4} = \sum_{n=1}^{\infty} \left(\sum_{i=-1}^{1} \frac{1}{F^2_{6n+2i}+4} \right) = -\frac{13}{40} + \frac{3\sqrt{5}}{16};$$
$$\sum_{n=2}^{\infty} \frac{1}{L^2_{2n}-20} = \sum_{n=1}^{\infty} \left(\sum_{i=-1}^{1} \frac{1}{L^2_{6n+2i}-20} \right) = \frac{43}{352} - \frac{3\sqrt{5}}{80};$$
$$\sum_{n=2}^{\infty} \frac{1}{F^2_{2n+1}-4} = \sum_{n=1}^{\infty} \left(\sum_{i=-1}^{1} \frac{1}{F^2_{6n+2i+1}-4} \right) = \frac{91}{192} - \frac{3\sqrt{5}}{16};$$
$$\sum_{n=2}^{\infty} \frac{1}{L^2_{2n+1}+20} = \sum_{n=1}^{\infty} \left(\sum_{i=-1}^{1} \frac{1}{L^2_{6n+2i+1}+20} \right) = -\frac{19}{252} + \frac{3\sqrt{5}}{80}.$$

Finally, we encourage the gibonacci enthusiasts to explore the gibonacci and Jacobsthal sums with $p = 5$, $k = 1 = r$; $p = 5$, $k = 2$, $r = 1$; and $p = 5$, $k = 2 = r$.

4. Acknowledgment

The authors are grateful to the reviewer for a careful reading of the article, and for his/her extraordinary patience, encouraging words, and constructive suggestions.

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MSC2020: Primary 11B37, 11B39, 11C08.

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