INFINITE SUMS INVOLVING POWERS OF GIBONACCI POLYNOMIALS

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ABSTRACT. We give formulas for infinite sums involving even (odd, respectively) powers of the gibonacci polynomials, unifying and extending some recent results on infinite sums involving low powers of the gibonacci polynomials.

1. INTRODUCTION

The *gibonacci polynomials* (short for generalized Fibonacci polynomials) $g_n(x)$ are defined by the second-order recurrence $g_{n+2}(x) = xg_{n+1}(x) + g_n(x)$, where x is an arbitrary integer variable, $g_0(x)$ and $g_1(x)$ are arbitrary integer polynomials, and $n \geq 0$. Let $g_0(x) = a = a(x)$ and $g_1(x) = b = b(x)$. If $a = 0$ and $b = 1$, then $g_n(x)$ is the nth *Fibonacci polynomial* $f_n(x)$ and if $a = 2$ and $b = x$, then $g_n(x)$ is the nth *Lucas polynomial* $\ell_n(x)$. In particular, $f_n(1)$ is the nth Fibonacci number F_n, and $\ell_n(1)$ is the nth Lucas number L_n, see [1,2]. The first four gibonacci polynomials are $g_0(x) = a$, $g_1(x) = b$, $g_2(x) = a + bx$, and $g_3(x) = ax + b(x^2 + 1)$. Note that there are two other closely related polynomials, one is the Pell polynomials $p_n(x)$, defined as $p_n(x) = f_n(2x)$, and the other is the Pell-Lucas polynomials $q_n(x)$, defined as $q_n(x) = \ell_n(2x)$ [2]. For brevity and convenience, we omit the argument x in the functional notation.

Koshy studied systematically properties of the gibonacci polynomials such as infinite sums of the squares and fourth powers of the gibonacci polynomials. Throughout the paper, let k be a positive integer; $r = 1, \ldots, k$;

$$L = \begin{cases} (k+1)/2, & \text{if } k \text{ is odd;} \\ k/2 + 1, & \text{otherwise;} \end{cases}$$

$$M = \begin{cases} (k+1)/2, & \text{if } k \text{ is odd;} \\ k/2, & \text{otherwise;} \end{cases}$$

$$s = \begin{cases} 2r - 1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise;} \end{cases}$$

$$t = \begin{cases} 2r, & \text{if } k \text{ is odd;} \\ 2r - 1, & \text{otherwise;} \end{cases}$$

and $\Delta = \sqrt{x^2 + 4}$. Koshy [5] proved the following formulas for infinite sums involving the squares of the gibonacci polynomials, which were also derived via graph-theoretic techniques in [6].

$$\sum_{n=L}^{\infty} \frac{f_{2k} f_{4n}}{[f_{2n}^2 - (-1)^k f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{f_s^2}, \tag{1.1}$$

$$\sum_{n=L}^{\infty} \frac{f_{2k} f_{4n}}{[\ell_{2n}^2 + (-1)^k \Delta^2 f_k^2]^2} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_s^2}, \tag{1.2}$$

$$\sum_{n=M}^{\infty} \frac{f_{2k} f_{4n+2}}{[f_{2n+1}^2 + (-1)^k f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{f_t^2}, \tag{1.3}$$

$$\sum_{n=M}^{\infty} \frac{f_{2k} f_{4n+2}}{[\ell_{2n+1}^2 - (-1)^k \Delta^2 f_k^2]^2} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_t^2}. \tag{1.4}$$

For infinite sums involving the fourth powers of the gibonacci polynomials, Koshy [7] also established the following formulas.

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} - 4(-1)^k f_{2k} f_{4n}}{[f_{2n}^2 - (-1)^k f_k^2]^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_s^4}, \tag{1.5}$$

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} + 4(-1)^k f_{2k} f_{4n}}{[\ell_{2n}^2 + (-1)^k \Delta^2 f_k^2]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_s^4}, \tag{1.6}$$

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} + 4(-1)^k f_{2k} f_{4n+2}}{[f_{2n+1}^2 + (-1)^k f_k^2]^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_t^4}, \tag{1.7}$$

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} - 4(-1)^k f_{2k} f_{4n+2}}{[\ell_{2n+1}^2 - (-1)^k \Delta^2 f_k^2]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_t^4}. \tag{1.8}$$

More results may be found in [3, 8–12]. Inspired by Koshy's results on infinite sums involving powers of the gibonacci polynomials, we establish some further properties on infinite sums involving powers of the gibonacci polynomials. The aim of the paper is twofold. First, we try to establish formulas on infinite sums involving even powers of the gibonacci polynomials by extending the above results as follows.

Theorem 1.1. *Let m be a positive integer. Then*

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{(k+1)j} f_{2k(m-j)} f_{4n(m-j)}}{[f_{2n}^2 - (-1)^k f_k^2]^{2m}} = \Delta^{2(m-1)} \sum_{r=1}^{k} \frac{1}{f_s^{2m}}.$$

Theorem 1.2. *Let m be a positive integer. Then*

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{kj} f_{2k(m-j)} f_{4n(m-j)}}{[\ell_{2n}^2 + (-1)^k \Delta^2 f_k^2]^{2m}} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_s^{2m}}.$$

Theorem 1.3. *Let m be a positive integer. Then*

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{kj} f_{2k(m-j)} f_{(4n+2)(m-j)}}{[f_{2n+1}^2 + (-1)^k f_k^2]^{2m}} = \Delta^{2(m-1)} \sum_{r=1}^{k} \frac{1}{f_t^{2m}}.$$

Theorem 1.4. *Let m be a positive integer. Then*

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{(k+1)j} f_{2k(m-j)} f_{(4n+2)(m-j)}}{[\ell_{2n+1}^2 - (-1)^k \Delta^2 f_k^2]^{2m}} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_t^{2m}}.$$

Letting $m = 1, 2$ in Theorem 1.1, we have (1.1) and (1.5). So, Theorem 1.1 extends (1.1) and (1.5). Similarly, Theorem 1.2 extends (1.2) and (1.6), Theorem 1.3 extends (1.3) and (1.7), and Theorem 1.4 extends (1.4) and (1.8).

Second, we give formulas involving infinite sums involving odd powers of the gibonacci polynomials as follows.

Theorem 1.5. *For a positive integer n and a nonnegative integer m with $n \geq k$,*

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} f_{2n(2m+1-2j)} \ell_{k(2m+1-2j)}}{(f_{2n}^2 + f_k^2)^{2m+1}} = \Delta^{2m} \sum_{r=1}^{k} \frac{1}{f_{2r-1}^{2m+1}} \text{ if } k \text{ is odd, and}$$

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \ell_{2n(2m+1-2j)} f_{k(2m+1-2j)}}{(f_{2n}^2 - f_k^2)^{2m+1}} = \Delta^{2m} \sum_{r=1}^{k} \frac{1}{f_{2r}^{2m+1}} \text{ if } k \text{ is even.}$$

Theorem 1.6. *For a positive integer n and a nonnegative integer m with $n \geq k$,*

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \ell_{2n(2m+1-2j)} \ell_{k(2m+1-2j)}}{(\ell_{2n}^2 - \Delta^2 f_k^2)^{2m+1}} = \sum_{r=1}^{k} \frac{1}{\ell_{2r-1}^{2m+1}} \text{ if } k \text{ is odd, and}$$

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} f_{2n(2m+1-2j)} f_{k(2m+1-2j)}}{(\ell_{2n}^2 + \Delta^2 f_k^2)^{2m+1}} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_{2r}^{2m+1}} \text{ if } k \text{ is even.}$$

Theorem 1.7. *For a positive integer n and a nonnegative integer m with $n \geq k$,*

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j f_{(2n+1)(2m+1-2j)} \ell_{k(2m+1-2j)}}{(f_{2n+1}^2 - f_k^2)^{2m+1}} = \Delta^{2m} \sum_{r=1}^{k} \frac{1}{f_{2r}^{2m+1}} \text{ if } k \text{ is odd, and}$$

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} \ell_{(2n+1)(2m+1-2j)} f_{k(2m+1-2j)}}{(f_{2n+1}^2 + f_k^2)^{2m+1}} = \Delta^{2m} \sum_{r=1}^{k} \frac{1}{f_{2r-1}^{2m+1}} \text{ if } k \text{ is even.}$$

Theorem 1.8. *For a positive integer n and a nonnegative integer m with $n \geq k$,*

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} \ell_{(2n+1)(2m+1-2j)} \ell_{k(2m+1-2j)}}{(\ell_{2n+1}^2 + \Delta^2 f_k^2)^{2m+1}} = \sum_{r=1}^{k} \frac{1}{\ell_{2r}^{2m+1}} \text{ if } k \text{ is odd, and}$$

$$\sum_{n=M}^{\infty} \frac{\sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j f_{(2n+1)(2m+1-2j)} f_{k(2m+1-2j)}}{(\ell_{2n+1}^2 - \Delta^2 f_k^2)^{2m+1}} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{\ell_{2r-1}^{2m+1}} \text{ if } k \text{ is even.}$$

The case $m = 0$ in Theorems 1.5, 1.6, 1.7, and 1.8 reduces to equations (5), (9), (7), and (11), respectively in [3].

As corollaries of Theorems 1.5–1.8, we derive a series of identities related to Fibonacci and Lucas polynomials (numbers, respectively).

2. Preliminaries

We need the following general properties of the gibonacci polynomials, where $g_n = f_n, \ell_n$.

Lemma 2.1. *[5] Let n be a positive integer with $n \geq k$. Then*

$$g_{n+k} g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k} \Delta^2 f_k^2, & \text{otherwise.} \end{cases}$$

We also need the following telescoping sums suitable for any positive integer power λ of $g(x)$, established in [5].

Lemma 2.2. *[5]*

$$\sum_{\substack{n=(k+1)/2 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{g_{2n-k}^{\lambda}} - \frac{1}{g_{2n+k}^{\lambda}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^{\lambda}}.$$

Lemma 2.3. *[5]*

$$\sum_{\substack{n=k/2+1\\ k\ even}}^{\infty}\left(\frac{1}{g_{2n-k}^{\lambda}}-\frac{1}{g_{2n+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 2.4. *[5]*

$$\sum_{\substack{n=(k+1)/2\\ k\ odd}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 2.5. *[5]*

$$\sum_{\substack{n=k/2\\ k\ even}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r-1}^{\lambda}}.$$

Fibonacci polynomials $f_n(x)$ and Lucas polynomials $\ell_n(x)$ can be calculated by the Binet-like formulas [2]: $f_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $\ell_n=\alpha^n+\beta^n$, respectively, where $\alpha=\frac{x+\sqrt{x^2+4}}{2}$ and $\beta=\frac{x-\sqrt{x^2+4}}{2}$. Clearly, $\alpha-\beta=\Delta$ and $\alpha\beta=-1$.

3. Sums Involving Even Powers of the Gibonacci Polynomials

In this section, we investigate sums involving even powers of the gibonacci polynomials. We need two key lemmas.

Lemma 3.1. *Let n and m be positive integers with $n\geq k$. Then*

$$\Delta^{2(m-1)}\left(f_{n+k}^{2m}-f_{n-k}^{2m}\right)=\sum_{j=0}^{m-1}\binom{2m}{j}(-1)^{(n+k+1)j}f_{2k(m-j)}f_{2n(m-j)}.$$

Proof. For $j=0,\dots,m-1$, let

$$\begin{aligned}A_j=&\binom{2m}{j}(-1)^j\alpha^{(n+k)(2m-j)}\beta^{(n+k)j}+\binom{2m}{2m-j}(-1)^{2m-j}\alpha^{(n+k)j}\beta^{(n+k)(2m-j)}\\&-\binom{2m}{j}(-1)^j\alpha^{(n-k)(2m-j)}\beta^{(n-k)j}-\binom{2m}{2m-j}(-1)^{2m-j}\alpha^{(n-k)j}\beta^{(n-k)(2m-j)}.\end{aligned}$$

Because $\alpha\beta=-1$, we have

$$\begin{aligned}A_j=&\binom{2m}{j}(-1)^j\Big(\alpha^{(n+k)(2m-j)}\beta^{(n+k)j}+\alpha^{(n+k)j}\beta^{(n+k)(2m-j)}\\&-\alpha^{(n-k)(2m-j)}\beta^{(n-k)j}-\alpha^{(n-k)j}\beta^{(n-k)(2m-j)}\Big)\\=&\binom{2m}{j}(-1)^j\Big(\alpha^{(n+k)(2m-2j)}\alpha^{(n+k)j}\beta^{(n+k)j}+\alpha^{(n+k)j}\beta^{(n+k)j}\beta^{(n+k)(2m-2j)}\\&-\alpha^{(n-k)(2m-2j)}\alpha^{(n-k)j}\beta^{(n-k)j}-\alpha^{(n-k)j}\beta^{(n-k)j}\beta^{(n-k)(2m-2j)}\Big)\\=&\binom{2m}{j}(-1)^j\Big[(-1)^{(n+k)j}\alpha^{(n+k)(2m-2j)}+(-1)^{(n+k)j}\beta^{(n+k)(2m-2j)}\\&-(-1)^{(n-k)j}(-1)^{2kj}\alpha^{(n-k)(2m-2j)}-(-1)^{(n-k)j}(-1)^{2kj}\beta^{(n-k)(2m-2j)}\Big]\\=&\binom{2m}{j}(-1)^{(n+k+1)j}\left(\alpha^{(n+k)(2m-2j)}+\beta^{(n+k)(2m-2j)}-\alpha^{(n-k)(2m-2j)}-\beta^{(n-k)(2m-2j)}\right)\end{aligned}$$

$$
\begin{aligned}
&= \binom{2m}{j}(-1)^{(n+k+1)j}\Big(\alpha^{(n+k)(2m-2j)} + \beta^{(n+k)(2m-2j)} \\
&\quad -\alpha^{(n-k)(2m-2j)}\alpha^{(2m-2j)k}\beta^{(2m-2j)k} - \beta^{(n-k)(2m-2j)}\beta^{(2m-2j)k}\alpha^{(2m-2j)k}\Big) \\
&= \binom{2m}{j}(-1)^{(n+k+1)j}\Big(\alpha^{(n+k)(2m-2j)} + \beta^{(n+k)(2m-2j)} \\
&\quad -\alpha^{(2m-2j)n}\beta^{(2m-2j)k} - \alpha^{(2m-2j)k}\beta^{(2m-2j)n}\Big) \\
&= \binom{2m}{j}(-1)^{(n+k+1)j}\Big(\alpha^{2k(m-j)} - \beta^{2k(m-j)}\Big)\Big(\alpha^{2n(m-j)} - \beta^{2n(m-j)}\Big) \\
&= \Delta^2\binom{2m}{j}(-1)^{(n+k+1)j} f_{2k(m-j)} f_{2n(m-j)},
\end{aligned}
$$

where the last equality follows from the Binet-like formula. Note also that

$$
\begin{aligned}
&\binom{2m}{m}(-1)^m\Big(\alpha^{(n+k)(2m-m)}\beta^{(n+k)m} - \alpha^{(n-k)(2m-m)}\beta^{(n-k)m}\Big) \\
&= \binom{2m}{m}(-1)^m\Big[(-1)^{(n+k)m} - (-1)^{(n-k)m}(-1)^{2km}\Big] \\
&= 0.
\end{aligned}
$$

By the Binet-like formula and the binomial theorem, we have

$$
\begin{aligned}
&\Delta^{2m}\left(f_{n+k}^{2m} - f_{n-k}^{2m}\right) \\
&= (\alpha-\beta)^{2m}\left(f_{n+k}^{2m} - f_{n-k}^{2m}\right) \\
&= \left(\alpha^{n+k} - \beta^{n+k}\right)^{2m} - \left(\alpha^{n-k} - \beta^{n-k}\right)^{2m} \\
&= \sum_{j=0}^{m-1} A_j + \binom{2m}{m}(-1)^m\Big(\alpha^{(n+k)(2m-m)}\beta^{(n+k)m} - \alpha^{(n-k)(2m-m)}\beta^{(n-k)m}\Big) \\
&= \sum_{j=0}^{m-1} A_j \\
&= \Delta^2\sum_{j=0}^{m-1}\binom{2m}{j}(-1)^{(n+k+1)j} f_{2k(m-j)} f_{2n(m-j)}. \qquad \square
\end{aligned}
$$

By a proof similar to that of Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let n and m be positive integers with $n \geq k$. Then*

$$
\ell_{n+k}^{2m} - \ell_{n-k}^{2m} = \Delta^2\sum_{j=0}^{m-1}\binom{2m}{j}(-1)^{(n+k)j} f_{2k(m-j)} f_{2n(m-j)}.
$$

Now, we are ready to prove Theorems 1.1–1.4.

Proof of Theorem 1.1. By Lemma 2.1, we have

$$
f_{2n}^2 - (-1)^k f_k^2 = f_{2n+k} f_{2n-k}.
$$

By Lemma 3.1, we have

$$\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{(k+1)j} f_{2k(m-j)} f_{4n(m-j)} = \Delta^{2(m-1)} \left(f_{2n+k}^{2m} - f_{2n-k}^{2m} \right).$$

Thus,

$$\begin{aligned} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{(k+1)j} f_{2k(m-j)} f_{4n(m-j)}}{[f_{2n}^2 - (-1)^k f_k^2]^{2m}} &= \Delta^{2(m-1)} \cdot \frac{f_{2n+k}^{2m} - f_{2n-k}^{2m}}{f_{2n+k}^{2m} f_{2n-k}^{2m}} \\ &= \Delta^{2(m-1)} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right). \end{aligned} \tag{3.1}$$

Summing both sides of (3.1) over all $n \in [L, \infty)$, we get

$$\sum_{n=L}^{\infty} \frac{\sum_{j=0}^{m-1} \binom{2m}{j} (-1)^{(k+1)j} f_{2k(m-j)} f_{4n(m-j)}}{[f_{2n}^2 - (-1)^k f_k^2]^{2m}} = \Delta^{2(m-1)} \sum_{n=L}^{\infty} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right).$$

If k is odd, then we have, by Lemma 2.2, that

$$\begin{aligned} \sum_{n=L}^{\infty} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right) &= \sum_{\substack{n=(k+1)/2 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right) \\ &= \sum_{r=1}^{k} \frac{1}{f_{2r-1}^{2m}} = \sum_{r=1}^{k} \frac{1}{f_s^{2m}}. \end{aligned}$$

If k is even, then we have, by Lemma 2.3, that

$$\begin{aligned} \sum_{n=L}^{\infty} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right) &= \sum_{\substack{n=k/2+1 \\ k \text{ even}}}^{\infty} \left(\frac{1}{f_{2n-k}^{2m}} - \frac{1}{f_{2n+k}^{2m}} \right) \\ &= \sum_{r=1}^{k} \frac{1}{f_{2r}^{2m}} = \sum_{r=1}^{k} \frac{1}{f_s^{2m}}. \end{aligned}$$

Now, the result follows readily. □

The proofs of Theorems 1.2–1.4 are similar to those of Theorem 1.1 (by Lemmas 2.1, 3.2, 2.2, and 2.3 for Theorem 1.2, Lemmas 2.1, 3.1, 2.4, and 2.5 for Theorem 1.3, and Lemmas 2.1, 3.2, 2.4, and 2.5 for Theorem 1.4) and are left as exercises for the interested reader.

4. Sums Involving Odd Powers of the Gibonacci Polynomials

We need some lemmas in the proofs.

Lemma 4.1. *Let n be a positive integer with $n \geq k$ and m be a nonnegative integer. Then*

$$\Delta^{2m} \left(f_{n+k}^{2m+1} - f_{n-k}^{2m+1} \right) = \begin{cases} \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^{nj} f_{n(2m+1-2j)} \ell_{k(2m+1-2j)}, & \text{if } k \text{ is odd;} \\ \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^{(n+1)j} \ell_{n(2m+1-2j)} f_{k(2m+1-2j)}, & \text{otherwise.} \end{cases}$$

Proof. For $j = 0, \ldots, m$, let

$$C_j = \binom{2m+1}{j} (-1)^j \alpha^{(n+k)(2m+1-j)} \beta^{(n+k)j}$$

$$+\binom{2m+1}{2m+1-j}(-1)^{2m+1-j}\alpha^{(n+k)j}\beta^{(n+k)(2m+1-j)}$$
$$-\binom{2m+1}{j}(-1)^{j}\alpha^{(n-k)(2m+1-j)}\beta^{(n-k)j}$$
$$-\binom{2m+1}{2m+1-j}(-1)^{2m+1-j}\alpha^{(n-k)j}\beta^{(n-k)(2m+1-j)}.$$

Because $\alpha\beta = -1$, we have

$$\begin{aligned}
C_j &= \binom{2m+1}{j}(-1)^{j}\Big(\alpha^{(n+k)(2m+1-j)}\beta^{(n+k)j} - \alpha^{(n+k)j}\beta^{(n+k)(2m+1-j)} \\
&\quad -\alpha^{(n-k)(2m+1-j)}\beta^{(n-k)j} + \alpha^{(n-k)j}\beta^{(n-k)(2m+1-j)}\Big) \\
&= \binom{2m+1}{j}(-1)^{(n+k+1)j}\Big(\alpha^{(n+k)(2m+1-2j)} - \beta^{(n+k)(2m+1-2j)} \\
&\quad -\alpha^{(n-k)(2m+1-2j)} + \beta^{(n-k)(2m+1-2j)}\Big).
\end{aligned}$$

Now, if k is odd, then, by the Binet-like formula, the binomial theorem, and $\alpha\beta = -1$, we have

$$\begin{aligned}
&\Delta^{2m}\left(f_{n+k}^{2m+1} - f_{n-k}^{2m+1}\right) \\
&= \frac{1}{\alpha-\beta}\left[(\alpha^{n+k}-\beta^{n+k})^{2m+1} - (\alpha^{n-k}-\beta^{n-k})^{2m+1}\right] \\
&= \frac{1}{\alpha-\beta}\sum_{j=0}^{m} C_j \\
&= \frac{1}{\alpha-\beta}\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{nj}\Big(\alpha^{(n+k)(2m+1-2j)} - \beta^{(n+k)(2m+1-2j)} \\
&\quad +\alpha^{(n-k)(2m+1-2j)}\alpha^{k(2m+1-2j)}\beta^{k(2m+1-2j)} \\
&\quad -\beta^{(n-k)(2m+1-2j)}\alpha^{k(2m+1-2j)}\beta^{k(2m+1-2j)}\Big) \\
&= \frac{1}{\alpha-\beta}\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{nj}\Big(\alpha^{(n+k)(2m+1-2j)} - \beta^{(n+k)(2m+1-2j)} \\
&\quad +\alpha^{n(2m+1-2j)}\beta^{k(2m+1-2j)} - \alpha^{k(2m+1-2j)}\beta^{n(2m+1-2j)}\Big) \\
&= \sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{nj}\left[\frac{1}{\alpha-\beta}\Big(\alpha^{n(2m+1-2j)} - \beta^{n(2m+1-2j)}\Big)\right] \\
&\quad \cdot\Big(\alpha^{k(2m+1-2j)} + \beta^{k(2m+1-2j)}\Big) \\
&= \sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{nj} f_{n(2m+1-2j)}\ell_{k(2m+1-2j)}.
\end{aligned}$$

Similarly, if k is even, then

$$\Delta^{2m}\left(f_{n+k}^{2m+1} - f_{n-k}^{2m+1}\right) = \sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{(n+1)j}\Big(\alpha^{n(2m+1-2j)} + \beta^{n(2m+1-2j)}\Big)$$

$$\cdot \left[\frac{1}{\alpha-\beta}\left(\alpha^{k(2m+1-2j)}-\beta^{k(2m+1-2j)}\right)\right]$$
$$=\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{(n+1)j}\ell_{n(2m+1-2j)}f_{k(2m+1-2j)},$$

as desired. □

By a proof similar to that of Lemma 4.1, we have the following lemma.

Lemma 4.2. *Let n be a positive integer with $n \geq k$ and m a nonnegative integer. Then*

$$\ell_{n+k}^{2m+1}-\ell_{n-k}^{2m+1}=\begin{cases}\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{(n+1)j}\ell_{n(2m+1-2j)}\ell_{k(2m+1-2j)}, & \text{if } k \text{ is odd;}\\ \Delta^2\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{nj}f_{n(2m+1-2j)}f_{k(2m+1-2j)}, & \text{otherwise.}\end{cases}$$

Now, we are ready to prove Theorems 1.5–1.8.

Proof of Theorem 1.5. Suppose first that k is odd. Using Lemmas 2.1 and 4.1, we get

$$\frac{\sum_{j=0}^{m}\binom{2m+1}{j}f_{2n(2m+1-2j)}\ell_{k(2m+1-2j)}}{(f_{2n}^2+f_k^2)^{2m+1}}=\Delta^{2m}\cdot\frac{f_{2n+k}^{2m+1}-f_{2n-k}^{2m+1}}{f_{2n+k}^{2m+1}f_{2n-k}^{2m+1}}$$
$$=\Delta^{2m}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right).$$

By Lemma 2.2, it follows that

$$\sum_{n=L}^{\infty}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right)=\sum_{\substack{n=(k+1)/2\\ k\text{ odd}}}^{\infty}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right)=\sum_{r=1}^{k}\frac{1}{f_{2r-1}^{2m+1}}.$$

So, the first formula follows.

Suppose next that k is even. By Lemmas 2.1 and 4.1, we get

$$\frac{\sum_{j=0}^{m}\binom{2m+1}{j}(-1)^{j}\ell_{2n(2m+1-2j)}f_{k(2m+1-2j)}}{(f_{2n}^2-f_k^2)^{2m+1}}=\Delta^{2m}\cdot\frac{f_{2n+k}^{2m+1}-f_{2n-k}^{2m+1}}{f_{2n+k}^{2m+1}f_{2n-k}^{2m+1}}$$
$$=\Delta^{2m}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right).$$

Using Lemma 2.3, we have

$$\sum_{n=L}^{\infty}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right)=\sum_{\substack{n=k/2+1\\ k\text{ even}}}^{\infty}\left(\frac{1}{f_{2n-k}^{2m+1}}-\frac{1}{f_{2n+k}^{2m+1}}\right)=\sum_{r=1}^{k}\frac{1}{f_{2r}^{2m+1}}.$$

Then the second formula follows. □

Setting $m=1$ and $k=1$ in the first formula and $m=1$ and $k=2$ in the second formula of Theorem 1.5, we have the following corollary.

Corollary 4.3.

$$\sum_{n=1}^{\infty}\frac{(x^3+3x)f_{6n}+3xf_{2n}}{(f_{2n}^2+1)^3}=x^2+4,$$
$$\sum_{n=2}^{\infty}\frac{x(x^4+4x^2+3)\ell_{6n}-3x\ell_{2n}}{(f_{2n}^2-x^2)^3}=\frac{(x^2+4)[(x^2+2)^3+1]}{x^3(x^2+2)^3}.$$

Setting $x = 1$ in Corollary 4.3, we have the following corollary.

Corollary 4.4.

$$\sum_{n=1}^{\infty} \frac{4F_{6n} + 3F_{2n}}{(F_{2n}^2 + 1)^3} = 5, \qquad \sum_{n=2}^{\infty} \frac{8L_{6n} - 3L_{2n}}{(F_{2n}^2 - 1)^3} = \frac{140}{27}.$$

The proofs to Theorems 1.6, 1.7, and 1.8 are similar to that of Theorem 1.5 (by Lemmas 2.1, 4.2, 2.2, and 2.3 for Theorem 1.6, Lemmas 2.1, 4.1, 2.4, and 2.5 for Theorem 1.7, and by Lemmas 2.1, 4.2, 2.4, and 2.5 for Theorem 1.8) and are left as exercises for the interested reader.

As immediate consequences of Theorem 1.6, we have the following corollaries.

Corollary 4.5.

$$\sum_{n=1}^{\infty} \frac{(x^3 + 3x)\ell_{6n} - 3x\ell_{2n}}{(\ell_{2n}^2 - \Delta^2)^3} = \frac{1}{x^3},$$

$$\sum_{n=2}^{\infty} \frac{x(x^4 + 4x^2 + 3)f_{6n} + 3xf_{2n}}{[\ell_{2n}^2 + (x^2 + 4)x^2]^3} = \frac{1}{x^2 + 4}\left(\frac{1}{(x^2 + 2)^3} + \frac{1}{(x^4 + 4x^2 + 2)^3}\right).$$

Corollary 4.6.

$$\sum_{n=1}^{\infty} \frac{4L_{6n} - 3L_{2n}}{(L_{2n}^2 - 5)^3} = 1, \qquad \sum_{n=2}^{\infty} \frac{8F_{6n} + 3F_{2n}}{(L_{2n}^2 + 5)^3} = \frac{74}{9261}.$$

From Theorem 1.7, we have the following corollaries.

Corollary 4.7.

$$\sum_{n=1}^{\infty} \frac{(x^3 + 3x)f_{6n+3} - 3xf_{2n+1}}{(f_{2n+1}^2 - 1)^3} = \frac{x^2 + 4}{x^3},$$

$$\sum_{n=1}^{\infty} \frac{x(x^4 + 4x^2 + 3)\ell_{6n+3} + 3x\ell_{2n+1}}{(f_{2n+1}^2 + x^2)^3} = \frac{(x^2 + 4)[(x^2 + 1)^3 + 1]}{(x^2 + 1)^3}.$$

Corollary 4.8.

$$\sum_{n=1}^{\infty} \frac{4F_{6n+3} - 3F_{2n+1}}{(F_{2n+1}^2 - 1)^3} = 5, \qquad \sum_{n=2}^{\infty} \frac{8L_{6n+3} + 3L_{2n+1}}{(F_{2n+1}^2 + 1)^3} = \frac{45}{8}.$$

From Theorem 1.8, we have the following corollaries.

Corollary 4.9.

$$\sum_{n=1}^{\infty} \frac{(x^3 + 3x)\ell_{6n+3} + 3x\ell_{2n+1}}{(\ell_{2n+1}^2 + x^2 + 4)^3} = \frac{1}{(x^2 + 2)^3},$$

$$\sum_{n=1}^{\infty} \frac{x(x^4 + 4x^2 + 3)f_{6n+3} - 3xf_{2n+1}}{[\ell_{2n+1}^2 - x^2(x^2 + 4)]^3} = \frac{(x^2 + 3)^3 + 1}{x^3(x^2 + 4)(x^2 + 3)^3}.$$

Corollary 4.10.

$$\sum_{n=1}^{\infty} \frac{4L_{6n+3} + 3L_{2n+1}}{(L_{2n+1}^2 + 5)^3} = \frac{1}{27}, \qquad \sum_{n=1}^{\infty} \frac{8F_{6n+3} - 3F_{2n+1}}{(L_{2n+1}^2 - 5)^3} = \frac{13}{64}.$$

5. Concluding Remark

Following Koshy's interesting work on infinite sums involving powers of the gibonacci polynomials, we give equalities on infinite sums involving powers of the gibonacci polynomials. In particular, unified equalities are found on infinite sums involving even powers of the gibonacci polynomials. For the first time, we investigate infinite sums involving odd powers of the gibonacci polynomials. It would be interesting to find graph-theoretical techniques as in [4, 6].

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