POLYNOMIAL VALUES WITH INTEGER COEFFICIENTS FOR THE GENERATING FUNCTIONS OF FIBONACCI POLYNOMIALS

YUJI TSUNO

Abstract. Fibonacci polynomials are generalizations of Fibonacci numbers, so it is natural to consider polynomial versions of the various results for Fibonacci numbers. According to Hong, Pongsriiam, and Bulawa and Lee, the generating function of the Fibonacci sequence in the domain of rational numbers, $f(t) = t/(1 - t - t^2)$, takes an integer value if and only if $t = F_k/F_{k+1}$ for some $k \in \mathbb{N}$ or $t = -F_{k+1}/F_k$ for some $k \in \mathbb{N}^+$, where F_k is the kth Fibonacci number. This study is built on their work by considering polynomial sequences that satisfy the recurrence relation $F_{i+2}(x) = axF_{i+1}(x)+bF_i(x)$ with initial values $(F_0(x), F_1(x)) = (0, 1)$, where a and b are positive integers such that $b|a$. As an application, for a square-free natural number $d \in \mathbb{N}$, we verify the results are of the same form as the above for the generating function of the sequence satisfying the recurrence relation $F_{i+2}(\sqrt{d}) = a\sqrt{d}F_{i+1}(\sqrt{d}) + bF_i(\sqrt{d})$ with initial values $(F_0(\sqrt{d}), F_1(\sqrt{d})) = (0, 1)$.

1. Introduction

Let a and b be positive integers. Bulawa and Lee [1] considered the sequence $\{F_i\}_{i\in\mathbb{N}}$ defined by the recurrence relation

$$F_{i+2} = aF_{i+1} + bF_i$$

and $F_0 = 0$ and $F_1 = 1$. The generating function is given as

$$f(t) = \frac{t}{1 - at - bt^2}.$$

They established the following necessary and sufficient conditions that should be applied when the rational values in the interval of convergence for the generating function $f(t)$ for the sequence $\{F_i\}_{i\in\mathbb{N}}$ are integers.

Theorem 1.1 (Bulawa and Lee [1]). *Let q be a rational number. Let us assume that b divides a and that q lies within the interval of convergence for the generating function $f(t)$. Then, $f(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{ \frac{F_{2i}}{F_{2i+1}} \right\}_{i\in\mathbb{N}}.$$

This result answers the conjecture developed by Hong [2].

Independent of Bulawa and Lee, Pongsriiam [3] obtained similar results under the conditions of $a = 1$ and $b = 1$.

In this study, their work is expanded for applicability to polynomials.

Let us define a polynomial sequence $\{F_i(x)\}_{i\in\mathbb{N}}$ given by the recurrence relation

$$F_{i+2}(x) = axF_{i+1}(x) + bF_i(x) \tag{1}$$

and $F_0(x) = 0$ and $F_1(x) = 1$. For example, $F_2(x) = ax$ and $F_3(x) = a^2x^2 + b$. The generating function is given by

$$f(x,t) = \sum_{i=0}^{\infty} F_i(x)t^i = \frac{t}{1 - axt - bt^2}.$$

This equation holds within the radius of convergence, but in this paper, we define $f(x,t) = t/(1 - axt - bt^2)$. Let us also define a polynomial sequence $\{L_i(x)\}_{i\in\mathbb{N}}$ given by the recurrence relations

$$L_{i+2}(x) = axL_{i+1}(x) + bL_i(x) \tag{2}$$

and $L_0(x) = 2$ and $L_1(x) = ax$. For example, $L_2(x) = a^2x^2 + 2b$ and $L_3(x) = a^3x^3 + 3abx$. The generating function is given by

$$l(x,t) = \sum_{i=0}^{\infty} L_i(x)t^i = \frac{2 - axt}{1 - axt - bt^2}.$$

This equation holds within the radius of convergence, but in this paper, we define $l(x,t) = (2 - axt)/(1 - axt - bt^2)$.

The main results are as follows.

Theorem 1.2. *Suppose that b divides a, and let $q(x) \in \mathbb{Q}(x)$ be a rational function over $\mathbb{Q}$. For the generating function $f(x,t)$, $f(x, q(x)) \in \mathbb{Z}[x]$ if and only if*

$$q(x) \in \left\{\frac{F_i(x)}{F_{i+1}(x)}\right\}_{i\in\mathbb{N}} \quad or \quad q(x) \in \left\{-\frac{F_{i+1}(x)}{bF_i(x)}\right\}_{i\in\mathbb{N}^+}.$$

Theorem 1.3. *Suppose that b divides a, and let $q(x) \in \mathbb{Q}(x)$ be a rational function over $\mathbb{Q}$. For the generating function $l(x,t)$, $l(x, q(x)) \in \mathbb{Z}[x]$ if and only if*

$$q(x) \in \left\{\frac{F_i(x)}{F_{i+1}(x)}, \frac{L_i(x)}{L_{i+1}(x)}, -\frac{L_{i+1}(x)}{bL_i(x)}\right\}_{i\in\mathbb{N}} \quad or \quad q(x) \in \left\{-\frac{F_{i+1}(x)}{bF_i(x)}\right\}_{i\in\mathbb{N}^+}.$$

Here, we remark that

$$\mathbb{Q}(x) = \left\{\frac{p_1(x)}{p_2(x)} \;\middle|\; p_1(x), p_2(x) \in \mathbb{Q}[x], \quad p_2(x) \neq 0\right\}$$

is the field of rational functions over $\mathbb{Q}$.

Let $d \in \mathbb{N}$ be a square-free natural number. We define a sequence $\{F_i(\sqrt{d})\}_{i\in\mathbb{N}}$ given by recurrence relation

$$F_{i+2}(\sqrt{d}) = a\sqrt{d}F_{i+1}(\sqrt{d}) + bF_i(\sqrt{d})$$

and $F_0(\sqrt{d}) = 0$ and $F_1(\sqrt{d}) = 1$. The generating function is given by

$$f(\sqrt{d},t) = \frac{t}{1 - a\sqrt{d}t - bt^2}.$$

We define a sequence $\{L_i(\sqrt{d})\}_{i\in\mathbb{N}}$ given by recurrence relations

$$L_{i+2}(\sqrt{d}) = a\sqrt{d}L_{i+1}(\sqrt{d}) + bL_i(\sqrt{d})$$

and $L_0(\sqrt{d}) = 2, L_1(\sqrt{d}) = a\sqrt{d}$. The generating function is given by

$$l(\sqrt{d},t) = \frac{2 - a\sqrt{d}t}{1 - a\sqrt{d}t - bt^2}.$$

Furthermore, the convergence radii of these generating functions are all

$$\frac{2}{a\sqrt{d}+\sqrt{a^2d+4b}}.$$

If $d \neq 1$, then in general we do not derive the same result as Theorem 1.2 and Theorem 1.3.

Indeed, assuming $a = 2$, $b = 1$, and $d = 2$, we obtain

$$f\left(\sqrt{d}, \frac{1}{2+\sqrt{d}}\right) = 2+\sqrt{d} \in \mathbb{Z}[\sqrt{d}].$$

However,

$$\frac{1}{2+\sqrt{d}} \notin \left\{\frac{F_i(\sqrt{d})}{F_{i+1}(\sqrt{d})}\right\}_{i\in\mathbb{N}} \quad \text{and} \quad \frac{1}{2+\sqrt{d}} \notin \left\{-\frac{F_{i+1}(\sqrt{d})}{bF_i(\sqrt{d})}\right\}_{i\in\mathbb{N}^+}.$$

Moreover, $1/(2+\sqrt{d})$ is within the radius of convergence of the generating function $f(\sqrt{d}, t)$. In addition, assuming $a = 1$, $b = 1$, and $d = 2$, we obtain

$$l\left(\sqrt{d}, \frac{6-5\sqrt{d}}{7}\right) = 16-10\sqrt{d} \in \mathbb{Z}[\sqrt{d}].$$

However,

$$\frac{6-5\sqrt{d}}{7} \notin \left\{\frac{F_i(\sqrt{d})}{F_{i+1}(\sqrt{d})}, \frac{L_i(\sqrt{d})}{L_{i+1}(\sqrt{d})}, -\frac{L_{i+1}(\sqrt{d})}{bL_i(\sqrt{d})}\right\}_{i\in\mathbb{N}} \quad \text{and} \quad \frac{6-5\sqrt{d}}{7} \notin \left\{-\frac{F_{i+1}(\sqrt{d})}{bF_i(\sqrt{d})}\right\}_{i\in\mathbb{N}^+}.$$

Moreover, $(6-5\sqrt{d})/7$ is within the radius of convergence of the generating function $l(\sqrt{d}, t)$.

If $d = 1$, then we have the following theorems.

Theorem 1.4. *Suppose that b divides a and $d = 1$. Let $q \in \mathbb{Q}$. For the generating function $f(t)$, we have $f(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{\frac{F_i}{F_{i+1}}\right\}_{i\in\mathbb{N}} \quad \text{or} \quad q \in \left\{-\frac{F_{i+1}}{bF_i}\right\}_{i\in\mathbb{N}^+},$$

where $f(t) = f(\sqrt{d}, t)$ and $F_i = F_i(\sqrt{d})$.

Theorem 1.5. *Suppose that b divides a and $d = 1$, and let $q \in \mathbb{Q}$ be a rational number. For the generating function $l(t)$, $l(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{\frac{F_i}{F_{i+1}}, \frac{L_i}{L_{i+1}}, -\frac{L_{i+1}}{bL_i}\right\}_{i\in\mathbb{N}} \quad \text{or} \quad q \in \left\{-\frac{F_{i+1}}{bF_i}\right\}_{i\in\mathbb{N}^+},$$

where $f(t) = f(\sqrt{d}, t)$, $F_i = F_i(\sqrt{d})$ and $L_i = L_i(\sqrt{d})$.

Focusing on the radii of convergence of the generating functions, we have the following results from Theorem 1.4 and Theorem 1.5.

Corollary 1.6 (= Theorem 1.1)**.** *Under the assumption of Theorem 1.4, let $q \in \mathbb{Q}$. We assume that q is in the interval of convergence of the generating function $f(t)$. Then, we have $f(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{\frac{F_{2i}}{F_{2i+1}}\right\}_{i\in\mathbb{N}}.$$

Corollary 1.7. *Suppose that b divides a, $d = 1$, and $a \neq 1$, let $q \in \mathbb{Q}$ be a rational number. We assume that q is in the interval of convergence of the generating function $l(t)$. For the generating function $l(t)$, $l(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{ \frac{F_{2i}}{F_{2i+1}}, \frac{L_{2i+1}}{L_{2i+2}} \right\}_{i \in \mathbb{N}}.$$

Corollary 1.8. *Suppose that b divides a, $d = 1$, and $a = 1$. Let $q \in \mathbb{Q}$ be a rational number. We assume that q is in the interval of convergence of the generating function $l(t)$. For the generating function $l(t)$, $l(q) \in \mathbb{Z}$ if and only if*

$$q \in \left\{ \frac{F_{2i}}{F_{2i+1}}, \frac{L_{2i+1}}{L_{2i+2}} \right\}_{i \in \mathbb{N}} \cup \left\{ -\frac{1}{2} \right\}.$$

Remark 1.9. *Corollary 1.6 and Corollary 1.8 are none other than those given by Bulawa and Lee [1, Theorem 1.1 and Theorem 1.5].*

2. Preliminaries

Before proceeding to the proof of the main results, the following equations and proposition should be understood.

Let

$$\alpha(x) = \frac{ax + \sqrt{a^2x^2 + 4b}}{2},$$
$$\beta(x) = \frac{ax - \sqrt{a^2x^2 + 4b}}{2}.$$

Then, it follows that

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \tag{3}$$

and

$$L_n(x) = \alpha(x)^n + \beta(x)^n. \tag{4}$$

Using equations (3) and (4), the following equations are obtained.

$$F_n(x)^2 - F_{n-1}(x)F_{n+1}(x) = (-b)^{n-1}, \tag{5}$$
$$L_n(x)^2 - L_{n-1}(x)L_{n+1}(x) = -(-b)^{n-1}(a^2x^2 + 4b), \tag{6}$$
$$F_{2n+1}(x) = L_{n+1}(x)F_n(x) + (-b)^n, \tag{7}$$
$$L_{2n+1}(x) = L_{n+1}(x)L_n(x) - (-b)^n ax, \tag{8}$$
$$L_{2n+1}(x) = (a^2x^2 + 4b)F_{n+1}(x)F_n(x) + (-b)^n ax, \tag{9}$$
$$F_{n+1}(x)L_n(x) = F_n(x)L_{n+1}(x) + 2(-b)^n, \tag{10}$$
$$F_{n+1}(x) = \frac{axF_n(x) + L_n(x)}{2}, \tag{11}$$
$$F_n(x) = \frac{-axF_{n+1}(x) + L_{n+1}(x)}{2b}, \tag{12}$$
$$L_{n+1}(x) = \frac{axL_n(x) + (a^2x^2 + 4b)F_n(x)}{2}, \quad \text{and} \tag{13}$$
$$L_n(x) = \frac{-axL_{n+1}(x) + (a^2x^2 + 4b)F_{n+1}(x)}{2b}. \tag{14}$$

The following proposition provides the most robust foundation for the proof of the main results.

Proposition 2.1. *Let $P(x), Q(x) \in \mathbb{Q}[x]$ be polynomials for which the highest-order coefficient is nonnegative. If*

$$P(x)^2 - (a^2x^2 + 4b)Q(x)^2 = 4(-b)^{r_0} \tag{$*$}$$

for some $r_0 \in \{0, 1\}$, then there exists a nonnegative integer n such that

$$b^{\lfloor \frac{n}{2} \rfloor} P(x) = L_n(x), \quad b^{\lfloor \frac{n}{2} \rfloor} Q(x) = F_n(x),$$

and $n \equiv r_0 \pmod 2$.

Proof. First, we define a map of the set of polynomial pairs with rational coefficients satisfying ($*$) to itself by

$$\Phi(P(x), Q(x)) = (\overline{P(x)}, \overline{Q(x)}),$$

where

$$\overline{P(x)} = \frac{-ax(a^2x^2 + 4b)Q(x) + (a^2x^2 + 2b)P(x)}{2b}$$

and

$$\overline{Q(x)} = \frac{(a^2x^2 + 2b)Q(x) - axP(x)}{2b}.$$

Also, $(\overline{P(x)}, \overline{Q(x)})$ satisfying equation ($*$) means $\overline{P(x)}$ and $\overline{Q(x)}$ satisfy

$$\overline{P(x)}^2 - (a^2x^2 + 4b)\overline{Q(x)}^2 = 4(-b)^{r_0}.$$

This is well-defined because $(\overline{P(x)}, \overline{Q(x)})$ satisfies equation ($*$). Moreover, the inverse map Φ^{-1} is given by

$$\Phi^{-1}(P(x), Q(x)) = (\underline{P(x)}, \underline{Q(x)}),$$

where

$$\underline{P(x)} = \frac{ax(a^2x^2 + 4b)Q(x) + (a^2x^2 + 2b)P(x)}{2b}$$

and

$$\underline{Q(x)} = \frac{(a^2x^2 + 2b)Q(x) + axP(x)}{2b}.$$

Moreover, we define $\Phi^{-1} \circ \Phi(P(x), Q(x)) = (\underline{\overline{P(x)}}, \underline{\overline{Q(x)}})$.

With the above preparation, we begin by considering the case $\deg Q(x) \leq 1$.

If $Q(x) = 0$, we have $P(x) = 2$ and $r_0 = 0$. Thus, we can obtain

$$b^{\lfloor \frac{0}{2} \rfloor} P(x) = L_0(x) \quad \text{and} \quad b^{\lfloor \frac{0}{2} \rfloor} Q(x) = F_0(x).$$

If $Q(x) \neq 0$ and $\deg Q(x) = 0$, then we have $P(x) = ax$, $Q(x) = 1$, and $r_0 = 1$ by matching coefficients of terms with equal degree. Thus, we can obtain

$$b^{\lfloor \frac{1}{2} \rfloor} P(x) = L_1(x) \quad \text{and} \quad b^{\lfloor \frac{1}{2} \rfloor} Q(x) = F_1(x).$$

If $\deg Q(x) = 1$, the method of undetermined coefficients gives us

$$P(x) = \frac{a^2x^2 + 2b}{b}, \quad Q(x) = \frac{ax}{b}, \quad \text{and} \quad r_0 = 0.$$

Thus, we can obtain the following results.

$$b^{\lfloor \frac{2}{2} \rfloor} P(x) = L_2(x) \quad \text{and} \quad b^{\lfloor \frac{2}{2} \rfloor} Q(x) = F_2(x).$$

Next, let us consider the case $\deg Q(x) \geq 2$.

We have $\deg P(x) = \deg Q(x) + 1$ because equation $(*)$ is satisfied.

Let $N = \deg P(x)$. Because equation $(*)$ is satisfied, the rational numbers $c_0, c_1, \ldots, c_N$, $d_0, d_1, \ldots, d_{N-1}$ exist such that $P(x) = c_0 x^N + c_1 x^{N-1} + \cdots + c_N$ and $Q(x) = d_0 x^{N-1} + d_1 x^{N-2} + \cdots + d_{N-1}$.

Then, we can obtain

$$c_0 = ad_0, \; c_1 = ad_1, \; ac_2 = a^2 d_2 + 2bd_0,$$

and

$$ac_3 = \begin{cases} a^2 d_3 + 2bd_1, & \text{if } \deg Q(x) > 2; \\ 2bd_1, & \text{if } \deg Q(x) = 2. \end{cases}$$

Moreover, if $\deg Q(x) > 2$,

$$\overline{P(x)} = \frac{(a^2 c_0 - a^3 d_0) x^{N+2}}{2b} + \frac{(a^2 c_1 - a^3 d_1) x^{N+1}}{2b} + \frac{(a^2 c_2 + 2bc_0 - a^3 d_2 - 4abd_0) x^N}{2b}$$
$$+ \frac{(a^2 c_3 + 2bc_1 - a^3 d_3 - 4abd_1) x^{N-1}}{2b} + \cdots.$$

If $\deg Q(x) = 2$,

$$\overline{P(x)} = \frac{(a^2 c_0 - a^3 d_0) x^5}{2b} + \frac{(a^2 c_1 - a^3 d_1) x^4}{2b} + \frac{(a^2 c_2 + 2bc_0 - a^3 d_2 - 4abd_0) x^3}{2b}$$
$$+ \frac{(a^2 c_3 + 2bc_1 - 4abd_1) x^2}{2b} + (c_2 - 2ad_2) x + c_3.$$

Therefore, we have

$$\deg \overline{P(x)} \leq \deg P(x) - 2$$

from the relationship between the coefficients of $P(x)$ and $Q(x)$. Moreover, we have

$$\deg \overline{Q(x)} \leq \deg Q(x) - 2.$$

Indeed, if $\deg \overline{P(x)} = 0$, then $\deg \overline{Q(x)} = 0$ because $(\overline{P(x)}, \overline{Q(x)})$ satisfies equation $(*)$.

If $\deg \overline{P(x)} > 0$, we have

$$\deg \overline{P(x)} = \deg \overline{Q(x)} + 1.$$

Therefore,

$$\deg \overline{Q(x)} = \deg \overline{P(x)} - 1 \leq \deg P(x) - 3 = \deg Q(x) - 2.$$

Moreover, we have

$$\Phi^{-1}(\overline{P(x)}, \overline{Q(x)}) = (\underline{\overline{P(x)}}, \underline{\overline{Q(x)}}) = (P(x), Q(x)).$$

Therefore, we see that the highest-order coefficient for $\overline{P(x)}$ (resp. $\overline{Q(x)}$) is a nonnegative rational number.

To show this, we first consider the case $\deg \overline{P(x)} = 0$. Then, we have $\overline{P(x)} = \pm 2$ and $\overline{Q(x)} = 0$ because $(\overline{P(x)}, \overline{Q(x)})$ satisfies equation $(*)$. If $\overline{P(x)} = -2$, the highest-order coefficient for $\underline{\overline{P(x)}} = P(x)$ is negative. This is a contradiction. Therefore, $\overline{P(x)} = 2$.

Next, we consider the case $\deg \overline{P(x)} \neq 0$. Let $\overline{c_0}$ (resp. $\overline{d_0}$) be the highest-order coefficient for $\overline{P(x)}$ (resp. $\overline{Q(x)}$). Then,

$$\overline{c_0} = \pm \overline{d_0} a$$

because $(\overline{P(x)}, \overline{Q(x)})$ satisfies equation $(*)$. If $\overline{c_0} = -\overline{d_0} a$, then

$$\begin{aligned} \underline{\overline{P(x)}} &= \frac{ax(a^2x^2+4b)\overline{Q(x)} + (a^2x^2+2b)\overline{P(x)}}{2b} \\ &= \frac{ax(a^2x^2+4b)(\overline{d_0}x^{\deg \overline{P(x)}-1} + \dots) + (a^2x^2+2b)(-\overline{d_0}ax^{\deg \overline{P(x)}} + \dots)}{2b}. \end{aligned}$$

Therefore,

$$\deg \overline{P(x)} + 2 > \deg \underline{\overline{P(x)}}.$$

Hence,

$$\deg \overline{P(x)} > \deg \underline{\overline{P(x)}} - 2 = \deg P(x) - 2.$$

This is a contradiction. Therefore, we have $\overline{c_0} = \overline{d_0} a$. From this, the signs of the highest-order coefficients of $\overline{P(x)}$ and $\overline{Q(x)}$ are equal. If the highest-order coefficient of $\overline{P(x)}$ is negative, then the highest-order coefficient of $\underline{\overline{P(x)}} = P(x)$ is negative. This is a contradiction. Therefore, the highest-order coefficient for $\overline{\overline{P(x)}}$ (resp. $\overline{Q(x)}$) is a nonnegative rational number.

Moreover, if $\deg Q(x) \geq 2$, we have

$$\deg \overline{P(x)} = \deg \underline{\overline{P(x)}} - 2 = \deg P(x) - 2$$

and

$$\deg \overline{Q(x)} = \deg \underline{\overline{Q(x)}} - 2 = \deg Q(x) - 2$$

because the highest-order coefficient for $\overline{P(x)}$ (resp. $\overline{Q(x)}$) is nonnegative.

Writing

$$\underbrace{\Phi \circ \cdots \circ \Phi}_{n}(P(x), Q(x)) = (\overset{n}{\overline{P(x)}}, \overset{n}{\overline{Q(x)}}),$$

we see that

$$\deg \overset{\lfloor \frac{N-1}{2} \rfloor}{\overline{P(x)}} \leq 2.$$

Finally, if there exists a nonnegative integer n such that

$$b^{\lfloor \frac{n}{2} \rfloor} \overline{P(x)} = L_n(x), b^{\lfloor \frac{n}{2} \rfloor} \overline{Q(x)} = F_n(x)$$

and $n \equiv r_0 \pmod 2$, then we have

$$b^{\lfloor \frac{n+2}{2} \rfloor} P(x) = L_{n+2}(x), b^{\lfloor \frac{n+2}{2} \rfloor} Q(x) = F_{n+2}(x)$$

and $n + 2 \equiv r_0 \pmod 2$ because

$$L_{m+2}(x) = \frac{ax(a^2x^2+4b)F_m(x) + (a^2x^2+2b)L_m(x)}{2}$$

and

$$F_{m+2}(x) = \frac{(a^2x^2+2b)F_m(x) + axL_m(x)}{2}$$

applies to any nonnegative integer m. If $\deg P(x) \leq 2$, we have

$$b^{\lfloor \frac{\deg P(x)}{2} \rfloor} P(x) = L_{\deg P(x)}(x), b^{\lfloor \frac{\deg P(x)}{2} \rfloor} Q(x) = F_{\deg P(x)}(x).$$

Therefore, this completes the proof.

□

The following proposition was obtained by Bulawa and Lee [1, Proposition 1.3], but we will prove it by the same proof method as Proposition 2.1.

Proposition 2.2. *Suppose that b divides a and b is square-free. Let $P, Q \in \mathbb{N}$. If*

$$P^2 - (a^2 + 4b)Q^2 = 4(-b), \tag{$**$}$$

then there exists a nonnegative integer n such that

$$b^n P = L_{2n+1}, b^n Q = F_{2n+1},$$

where $L_{2n+1} = L_{2n+1}(1)$ and $F_{2n+1} = F_{2n+1}(1)$.

Proof. First, we define a map of the set of integer pairs satisfying equation $(**)$ to itself by

$$\Phi(P, Q) = (\overline{P}, \overline{Q}),$$

where

$$\overline{P} = \frac{-a(a^2 + 4b)Q + (a^2 + 2b)P}{2b}$$

and

$$\overline{Q} = \frac{(a^2 + 2b)Q - aP}{2b}.$$

Also, $(\overline{P}, \overline{Q})$ satisfying equation $(**)$ means $\overline{P}$ and $\overline{Q}$ satisfy

$$\overline{P}^2 - (a^2 + 4b)\overline{Q}^2 = 4(-b).$$

We have $P^2 - (aQ)^2 \in 4\mathbb{Z}$ because (P, Q) satisfies equation $(**)$. From this, we obtain $P - aQ \in 2\mathbb{Z}$. Therefore, $\overline{P}, \overline{Q} \in \mathbb{Z}$. Hence, Φ is well-defined because $(\overline{P}, \overline{Q})$ satisfies equation $(**)$

Moreover, the inverse map Φ^{-1} is given by

$$\Phi^{-1}(P, Q) = (\underline{P}, \underline{Q}),$$

where

$$\underline{P} = \frac{a(a^2 + 4b)Q + (a^2 + 2b)P}{2b}$$

and

$$\underline{Q} = \frac{(a^2 + 2b)Q + aP}{2b}.$$

Furthermore, we prepare two maps.

Subsequently, we define a map of the set of integer pairs satisfying equation $(**)$ to the set of integer pairs satisfying equation

$$P^2 - (a^2 + 4b)Q^2 = 4 \tag{$***$}$$

by

$$\hat{\Phi}(P, Q) = (\hat{P}, \hat{Q}),$$

where

$$\hat{P} = \frac{-aP + (a^2 + 4b)Q}{2b}$$

and

$$\hat{Q} = \frac{-aQ + P}{2b}.$$

Also, $(\hat{P},\hat{Q})$ satisfying equation $(***)$ means $\hat{P}$ and $\hat{Q}$ satisfy

$$\hat{P}^2-(a^2+4b)\hat{Q}^2=4.$$

We have $P^2-a^2Q^2\in 4b\mathbb{Z}$ because (P,Q) satisfies equation $(**)$. From this, we obtain $P-aQ\in 2b\mathbb{Z}$. Indeed, we have $P\in b\mathbb{Z}$ because $P^2\in b\mathbb{Z}$ and b is square-free. Therefore, we obtain $P-aQ, P+aQ\in b\mathbb{Z}$. Hence, there exist $l_1,l_2\in\mathbb{Z}$ such that $P-aQ=bl_1$ and $P+aQ=bl_2$. Moreover, there exists $l_3\in\mathbb{Z}$ such that $bl_1bl_2=(P-aQ)(P+aQ)=P^2-a^2Q^2=4bl_3$. If $l_3=0$, we have that $l_1=0$ or $l_2=0$. Therefore, $P-aQ\in 2b\mathbb{Z}$. If $l_3\neq 0$, we have $l_1\in 2\mathbb{Z}$ or $l_2\in 2\mathbb{Z}$. Therefore, $P-aQ\in 2b\mathbb{Z}$.

Therefore, $\hat{P},\hat{Q}\in\mathbb{Z}$. Hence, $\hat{\Phi}$ is well-defined because $(\hat{P},\hat{Q})$ satisfies equation $(***)$.

If $P>0, Q>1$, then

$$\{(a^2+4b)Q\}^2-(aP)^2=4a^2bQ^2+16b^2Q^2+4a^2b>0$$

and

$$P^2-a^2Q^2=4bQ^2-4b>0$$

by $P^2=(a^2+4b)Q^2-4b$ because (P,Q) satisfies equation $(**)$.

Therefore,

$$\hat{P}=\frac{-aP+(a^2+4b)Q}{2b}>0 \text{ and } \hat{Q}=\frac{-aQ+P}{2b}>0.$$

Hence, $\hat{P}\geq 1$ and $\hat{Q}\geq 1$ because $\hat{P},\hat{Q}\in\mathbb{Z}$.

Finally, we define a map of the set of integer pairs satisfying equation $(***)$ to the set of integer pairs satisfying equation $(**)$ by

$$\check{\Phi}(P,Q)=(\check{P},\check{Q}),$$

where

$$\check{P}=\frac{-aP+(a^2+4b)Q}{2} \text{ and } \check{Q}=\frac{-aQ+P}{2}.$$

Also, $(\check{P},\check{Q})$ satisfying equation $(**)$ means $(\check{P},\check{Q})$ satisfies the equation $(**)$, where $\check{P}$ and $\check{Q}$ are substituted for P and Q, respectively. We have $P-aQ\in 2\mathbb{Z}$ because (P,Q) satisfies equation $(***)$. Therefore, $\check{P},\check{Q}\in\mathbb{Z}$. Hence, $\check{\Phi}$ is well-defined as $(\check{P},\check{Q})$ satisfies equation $(**)$.

If, $P>0, Q\geq 1$, then

$$\{(a^2+4b)Q\}^2-(aP)^2=4a^2bQ^2+16b^2Q^2-4a^2>0$$

and

$$P^2-(aQ)^2=4bQ^2+4>0$$

by $P^2=(a^2+4b)Q^2+4$ because (P,Q) satisfies equation $(***)$.

Therefore,

$$\check{P}=\frac{-aP+(a^2+4b)Q}{2}>0 \text{ and } \check{Q}=\frac{-aQ+P}{2}>0.$$

Hence, $\check{P}\geq 1$ and $\check{Q}\geq 1$ because $\check{P},\check{Q}\in\mathbb{Z}$.

Using the above preparation, let us consider the case $Q\leq 1$. Then, we have

$$P=L_1 \text{ and } Q=F_1.$$

Next, we consider the case $Q>1$. (Hence, $P>0$ because (P,Q) satisfies equation $(**)$.) Taking the image $\Phi(P,Q)=(\overline{P},\overline{Q})$ by Φ, we have $\overline{P},\overline{Q}>0$ because $P>0$ and $\Phi=\check{\Phi}\circ\hat{\Phi}$.

Moreover, we obtain

$$Q - \overline{Q} = \frac{a\sqrt{(a^2+4b)Q^2-4b} - a^2Q}{2b} > 0.$$

In addition,

$$Q - \overline{Q} \geq 1$$

because $Q, \overline{Q} \in \mathbb{N}$. Writing

$$\underbrace{\Phi \circ \cdots \circ \Phi}_{n}(P, Q) = (\overset{n}{\overline{P}}, \overset{n}{\overline{Q}}),$$

we see that there exists a positive integer m such that $\overset{m}{\overline{Q}} \leq 1$ Therefore, the proof is completed in the same way as Proposition 2.1. □

We have the following proposition from Proposition 2.2.

Proposition 2.3. *Suppose that b divides a. Let $P, Q \in \mathbb{N}$. If*

$$P^2 - (a^2+4b)Q^2 = 4(-b), \tag{$**$}$$

then there exists a nonnegative integer n such that

$$b^nP = L_{2n+1} \quad \text{and} \quad b^nQ = F_{2n+1},$$

where $L_{2n+1} = L_{2n+1}(1)$ and $F_{2n+1} = F_{2n+1}(1)$.

Proof. Let c be the largest positive integer such that $b \in c^2\mathbb{Z}$. Put $\tilde{b} = b/c^2$ and $\tilde{a} = a/c$. We define a sequence $\{\tilde{F}_i\}_{i\in\mathbb{N}}$ given by the recurrence relation

$$\tilde{F}_{i+2} = \tilde{a}\tilde{F}_{i+1} + \tilde{b}\tilde{F}_i$$

and $\tilde{F}_0 = 0$ and $\tilde{F}_1 = 1$. Moreover, we define a sequence $\{\tilde{L}_i\}_{i\in\mathbb{N}}$ given by the recurrence relation

$$\tilde{L}_{i+2} = \tilde{a}\tilde{L}_{i+1} + \tilde{b}\tilde{L}_i$$

and $\tilde{L}_0 = 2$ and $\tilde{L}_1 = \tilde{a}$.

By equations (3) and (4), it follows that

$$F_i = F_i(1) = \frac{\alpha(1)^i - \beta(1)^i}{\alpha(1) - \beta(1)}$$

and

$$L_i = L_i(1) = \alpha(1)^i + \beta(1)^i.$$

Moreover, we obtain

$$\tilde{F}_i = \frac{(\alpha(1)/c)^i - (\beta(1)/c)^i}{(\alpha(1)-\beta(1))/c}$$

and

$$\tilde{L}_i = \left(\frac{\alpha(1)}{c}\right)^i + \left(\frac{\beta(1)}{c}\right)^i.$$

Therefore,

$$F_i = c^{i-1}\tilde{F}_i \ (i \geq 1)$$

and

$$L_i = c^i\tilde{L}_i \ (i \geq 1).$$

On the other hand, $P/c \in \mathbb{N}$ and P/c and Q satisfy

$$\left(\frac{P}{c}\right)^2 - (\tilde{a}^2 + 4\tilde{b})Q^2 = -4\tilde{b}$$

Therefore, by Proposition 2.2, there exists a nonnegative integer n such that

$$\tilde{b}^n \frac{P}{c} = L_{2n+1}^{\sim}, \tilde{b}^n Q = F_{2n+1}^{\sim}$$

Hence, we obtain

$$b^n P = L_{2n+1}, b^n Q = F_{2n+1}.$$

□

The argument for obtaining Proposition 2.3 from Proposition 2.2 is similar to the argument in [1, Proposition 1.4].

3. Resulting Proofs

Like [4], the main results are demonstrated by applying Proposition 2.1, Proposition 2.3, and (1) to (14).

3.1. **Proof of Theorem 1.2.** First, we suppose that

$$q(x) = \frac{F_i(x)}{F_{i+1}(x)} \ (i \in \mathbb{N})$$

or

$$q(x) = -\frac{F_{i+1}(x)}{bF_i(x)} \ (i \in \mathbb{N}^+).$$

Then, we show that $f(x, q(x)) \in \mathbb{Z}[x]$. If $i = 0$, the result is evident. However, if $i > 0$, by using (1) and (5), we obtain

$$f\left(x, \frac{F_i(x)}{F_{i+1}(x)}\right) = \frac{F_i(x)F_{i+1}(x)}{F_{i+1}(x)^2 - (axF_{i+1}(x) + bF_i(x))F_i(x)}$$

$$\overset{(1)}{=} \frac{F_i(x)F_{i+1}(x)}{F_{i+1}(x)^2 - F_{i+2}(x)F_i(x)} \overset{(5)}{=} \frac{F_i(x)F_{i+1}(x)}{(-b)^i}$$

$$f\left(x, -\frac{F_{i+1}(x)}{bF_i(x)}\right) = \frac{-bF_i(x)F_{i+1}(x)}{bF_i(x)(axF_{i+1}(x) + bF_i(x)) - bF_{i+1}(x)^2}$$

$$\overset{(1)}{=} \frac{-bF_i(x)F_{i+1}(x)}{bF_i(x)F_{i+2}(x) - bF_{i+1}(x)^2} \overset{(5)}{=} \frac{F_i(x)F_{i+1}(x)}{(-b)^i}.$$

Moreover, we have $F_i(x) \in b^{\lfloor \frac{i}{2} \rfloor}\mathbb{Z}[x]$ $(i \in \mathbb{N})$ from the recurrence relation described by equation (1). Indeed, $F_0(x) \in b^{\lfloor \frac{0}{2} \rfloor}\mathbb{Z}[x]$ and $F_1(x) \in b^{\lfloor \frac{1}{2} \rfloor}\mathbb{Z}[x]$. If $F_k(x) \in b^{\lfloor \frac{k}{2} \rfloor}\mathbb{Z}[x]$ and $F_{k+1}(x) \in b^{\lfloor \frac{k+1}{2} \rfloor}\mathbb{Z}[x]$, then $F_{k+2}(x) \in b^{\lfloor \frac{k+2}{2} \rfloor}\mathbb{Z}[x]$ by equation (1). Therefore, we have $F_i(x) \in b^{\lfloor \frac{i}{2} \rfloor}\mathbb{Z}[x]$ by mathematical induction. Hence, $f(x, q(x)) \in \mathbb{Z}[x]$.

Next, we suppose that $f(x, q(x)) = k(x)$ ($k(x)$ is a polynomial over $\mathbb{Z}$) for some rational function $q(x) \in \mathbb{Q}(x)$. We will show that

$$q(x) \in \left\{\frac{F_i(x)}{F_{i+1}(x)}\right\}_{i \in \mathbb{N}} \quad \text{or} \quad q(x) \in \left\{-\frac{F_{i+1}(x)}{bF_i(x)}\right\}_{i \in \mathbb{N}^+}.$$

If $k(x) = 0$, then
$$\frac{q(x)}{1 - axq(x) - bq(x)^2} = 0.$$
Hence,
$$q(x) = 0 = \frac{F_0(x)}{F_1(x)}.$$
If $k(x) \neq 0$, then
$$\frac{q(x)}{1 - axq(x) - bq(x)^2} = k(x).$$
Hence,
$$bk(x)q(x)^2 + (axk(x) + 1)q(x) - k(x) = 0.$$
Furthermore,
$$q(x) = \frac{-(axk(x) + 1) \pm \sqrt{(axk(x) + 1)^2 + 4bk(x)^2}}{2bk(x)}.$$
Here, because $q(x)$ is a rational function over $\mathbb{Q}$, there exists a polynomial $M(x) \in \mathbb{Z}[x]$ for which the highest-order coefficient is nonnegative such that
$$(axk(x) + 1)^2 + 4bk(x)^2 = M(x)^2.$$
This allows us to obtain
$$\{(a^2x^2 + 4b)k(x) + ax\}^2 - (a^2x^2 + 4b)M(x)^2 = 4(-b).$$
Thus, according to Proposition 2.1, there exists a nonnegative integer n such that
$$M(x) = \frac{F_{2n+1}(x)}{b^n}, (a^2x^2 + 4b)k(x) + ax = \pm\frac{L_{2n+1}(x)}{b^n}.$$
From equation (9),
$$(a^2x^2 + 4b)k(x) + ax = \pm\frac{(a^2x^2 + 4b)F_{n+1}(x)F_n(x) + (-b)^n ax}{b^n}.$$
Because $k(x) \in \mathbb{Z}[x]$ and $\frac{F_n(x)F_{n+1}(x)}{b^n} \in \mathbb{Z}[x]$, this means that
$$(a^2x^2 + 4b)k(x) + ax = \frac{L_{2n+1}(x)}{(-b)^n}$$
for each $n \in \mathbb{N}^+$ given that $k(x) \neq 0$. Additionally, by using equation (9),
$$k(x) = \frac{F_n(x)F_{n+1}(x)}{(-b)^n}.$$
Consequently, we obtain
$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n + (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \ (n \geq 1) \qquad (A)$$
or
$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n - (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \ (n \geq 1). \qquad (B)$$
Regarding (A) and (B), by using equations (7), (10), (11), and (12), we obtain
$$q(x) \in \left\{\frac{F_i(x)}{F_{i+1}(x)}\right\}_{i\in\mathbb{N}} \quad \text{or} \quad q(x) \in \left\{-\frac{F_{i+1}(x)}{bF_i(x)}\right\}_{i\in\mathbb{N}^+}.$$
If n is even, for (A),

$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n + (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \overset{(7)}{=} \frac{-axF_{n+1}(x) + L_{n+1}(x)}{2bF_{n+1}(x)} \overset{(12)}{=} \frac{F_n(x)}{F_{n+1}(x)}.$$

If n is odd, for (A),

$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n + (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \overset{(7)(10)}{=} \frac{-axF_n(x) - L_n(x)}{2bF_n(x)} \overset{(11)}{=} -\frac{F_{n+1}(x)}{bF_n(x)}.$$

If n is even, for (B),

$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n - (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \overset{(7)(10)}{=} \frac{-axF_n(x) - L_n(x)}{2bF_n(x)} \overset{(11)}{=} -\frac{F_{n+1}(x)}{bF_n(x)}.$$

If n is odd, for (B),

$$q(x) = \frac{-axF_n(x)F_{n+1}(x) - (-b)^n - (-1)^n F_{2n+1}(x)}{2bF_n(x)F_{n+1}(x)} \overset{(7)}{=} \frac{-axF_{n+1}(x) + L_{n+1}(x)}{2bF_{n+1}(x)} \overset{(12)}{=} \frac{F_n(x)}{F_{n+1}(x)}.$$

3.2. **Proof of Theorem 1.3.** First, we suppose that

$$q(x) = \frac{F_i(x)}{F_{i+1}(x)}, \frac{L_i(x)}{L_{i+1}(x)}, -\frac{L_{i+1}(x)}{bL_i(x)} \ (i \in \mathbb{N}) \quad \text{or} \quad q(x) = -\frac{F_{i+1}(x)}{bF_i(x)} \ (i \in \mathbb{N}^+).$$

Then, we show that $l(x, q(x)) \in \mathbb{Z}[x]$

If $i = 0$, the result is evident. However, if $i > 0$, by using equations (1), (2), (5), (6), (13), and (14), we obtain

$$l\left(x, \frac{F_i(x)}{F_{i+1}(x)}\right) = \frac{2F_{i+1}(x)^2 - axF_i(x)F_{i+1}(x)}{F_{i+1}(x)^2 - F_i(x)(axF_{i+1}(x) + bF_i(x))} \overset{(1)}{=} \frac{2F_{i+1}(x)^2 - axF_i(x)F_{i+1}(x)}{F_{i+1}(x)^2 - F_i(x)F_{i+2}(x)} \overset{(5)}{=} \frac{2F_{i+1}(x)^2 - axF_i(x)F_{i+1}(x)}{(-b)^i}$$

$$l\left(x, \frac{L_i(x)}{L_{i+1}(x)}\right) = \frac{2L_{i+1}(x)^2 - axL_i(x)L_{i+1}(x)}{L_{i+1}(x)^2 - L_i(x)(axL_{i+1}(x) + bL_i(x))} \overset{(2)}{=} \frac{2L_{i+1}(x)^2 - axL_i(x)L_{i+1}(x)}{L_{i+1}(x)^2 - L_i(x)L_{i+2}(x)} \overset{(6)}{=} \frac{L_{i+1}(x)(2L_{i+1}(x) - axL_i(x))}{-(-b)^i(a^2x^2 + 4b)} \overset{(13)}{=} \frac{L_{i+1}(x)F_i(x)}{-(-b)^i}$$

$$l\left(x, -\frac{L_{i+1}(x)}{bL_i(x)}\right) = \frac{L_i(x)(2bL_i(x) + axL_{i+1}(x))}{L_i(x)(axL_{i+1}(x) + bL_i(x)) - L_{i+1}(x)^2} \overset{(2)}{=} \frac{L_i(x)(2bL_i(x) + axL_{i+1}(x))}{L_i(x)L_{i+2}(x) - L_{i+1}(x)^2} \overset{(6)(14)}{=} \frac{L_i(x)F_{i+1}(x)}{(-b)^i}$$

$$l\left(x, -\frac{F_{i+1}(x)}{bF_i(x)}\right) = \frac{F_i(x)(2bF_i(x) + axF_{i+1}(x))}{F_i(x)(bF_i(x) + axF_{i+1}(x)) - F_{i+1}(x)^2}$$

$$\overset{(1)(5)}{=} \frac{F_i(x)(2bF_i(x)+axF_{i+1}(x))}{F_i(x)F_{i+2}(x)-F_{i+1}(x)^2} = \frac{F_i(x)(2bF_i(x)+axF_{i+1}(x))}{-(-b)^i}.$$

Using an induction similar to the discussion in the proof of Theorem 1.2, we obtain that $F_i(x), L_i(x) \in b^{\lfloor \frac{i}{2} \rfloor}\mathbb{Z}[x]$ from the recurrence relations described by equations (1) and (2),

$$l\left(x, \frac{F_i(x)}{F_{i+1}(x)}\right),\quad l\left(x, \frac{L_i(x)}{L_{i+1}(x)}\right),$$

$$l\left(x, -\frac{L_{i+1}(x)}{bL_i(x)}\right) \in \mathbb{Z}[x]\ (i \in \mathbb{N}), \quad \text{and} \quad l\left(x, -\frac{F_{i+1}(x)}{bF_i(x)}\right) \in \mathbb{Z}[x]\ (i \in \mathbb{N}^+).$$

Next, we suppose that $l(x, q(x)) = k(x)$ ($k(x)$ is a polynomial over $\mathbb{Z}$) for some rational function $q(x) \in \mathbb{Q}(x)$. Then, we show that

$$q(x) \in \left\{\frac{F_i(x)}{F_{i+1}(x)}, \frac{L_i(x)}{L_{i+1}(x)}, -\frac{L_{i+1}(x)}{bL_i(x)}\right\}_{i\in\mathbb{N}} \quad \text{or} \quad q(x) \in \left\{-\frac{F_{i+1}(x)}{bF_i(x)}\right\}_{i\in\mathbb{N}^+}.$$

If $k(x) = 0$, then

$$\frac{2-axq(x)}{1-axq(x)-bq(x)^2} = 0.$$

Therefore,

$$q(x) = \frac{2}{ax} = \frac{L_0(x)}{L_1(x)}.$$

Alternatively, if $k(x) \neq 0$, then

$$\frac{2-axq(x)}{1-axq(x)-bq(x)^2} = k(x).$$

Hence,

$$bk(x)q(x)^2 + ax(k(x)-1)q(x) + 2 - k(x) = 0.$$

Therefore,

$$q(x) = \frac{-ax(k(x)-1) \pm \sqrt{a^2x^2(k(x)-1)^2 - 4bk(x)(2-k(x))}}{2bk(x)}.$$

Here, because $q(x)$ is a rational function over $\mathbb{Q}$, there exists a polynomial $M(x) \in \mathbb{Q}[x]$ for which the highest-order coefficient is nonnegative such that

$$a^2x^2(k(x)-1)^2 - 4bk(x)(2-k(x)) = M(x)^2.$$

Then, we obtain

$$M(x)^2 - (a^2x^2+4b)(k(x)-1)^2 = 4(-b).$$

Thus, according to Proposition 2.1, there exists a nonnegative integer n such that

$$M(x) = \frac{L_{2n+1}(x)}{b^n} \quad \text{and} \quad k(x) - 1 = \pm\frac{F_{2n+1}(x)}{b^n}.$$

Hence, we obtain

$$q(x) = \frac{-axF_{2n+1}(x) + L_{2n+1}(x)}{2b(F_{2n+1}(x)+b^n)}\ (n \geq 0), \tag{C}$$

$$q(x) = \frac{-axF_{2n+1}(x) - L_{2n+1}(x)}{2b(F_{2n+1}(x)+b^n)}\ (n \geq 0), \tag{D}$$

$$q(x) = \frac{axF_{2n+1}(x) + L_{2n+1}(x)}{2b(-F_{2n+1}(x)+b^n)}\ (n \geq 1), \tag{E}$$

or

$$q(x) = \frac{axF_{2n+1}(x) - L_{2n+1}(x)}{2b(-F_{2n+1}(x) + b^n)} \ (n \geq 1). \tag{F}$$

In cases (C) to (F), using equations (7), (9), (10), (13), and (14) gives us

$$q(x) \in \left\{ \frac{F_i(x)}{F_{i+1}(x)}, \frac{L_i(x)}{L_{i+1}(x)}, -\frac{L_{i+1}(x)}{bL_i(x)} \right\}_{i\in\mathbb{N}} \quad \text{or} \quad q(x) \in \left\{ -\frac{F_{i+1}(x)}{bF_i(x)} \right\}_{i\in\mathbb{N}^+}.$$

Then, if n is even, for (C),

$$q(x) = \frac{-axF_{2n+1}(x) + L_{2n+1}(x)}{2b(F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{-axL_{n+1}(x)F_n(x) + (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{2bF_{n+1}(x)L_n(x)}$$
$$\overset{(14)}{=} \frac{F_n(x)}{F_{n+1}(x)}.$$

If n is odd, for (C),

$$q(x) = \frac{-axF_{2n+1}(x) + L_{2n+1}(x)}{2b(F_{2n+1}(x) + b^n)} \overset{(7)(9)}{=} \frac{-axL_{n+1}(x)F_n(x) + (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{2bL_{n+1}(x)F_n(x)}$$
$$\overset{(14)}{=} \frac{L_n(x)}{L_{n+1}(x)}.$$

If n is even, for (D),

$$q(x) = \frac{-axF_{2n+1}(x) - L_{2n+1}(x)}{2b(F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{-axL_n(x) - (a^2x^2 + 4b)F_n(x)}{2bL_n(x)}$$
$$\overset{(13)}{=} -\frac{L_{n+1}(x)}{bL_n(x)}.$$

If n is odd, for (D),

$$q(x) = \frac{-axF_{2n+1}(x) - L_{2n+1}(x)}{2b(F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{-axF_{n+1}(x)L_n(x) - (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{2bL_{n+1}(x)F_n(x)}$$
$$\overset{(13)}{=} -\frac{F_{n+1}(x)}{bF_n(x)}.$$

If n is even, for (E),

$$q(x) = \frac{axF_{2n+1}(x) + L_{2n+1}(x)}{2b(-F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{axF_{n+1}(x)L_n(x) + (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{-2bL_{n+1}(x)F_n(x)}$$
$$\overset{(13)}{=} -\frac{F_{n+1}(x)}{bF_n(x)}.$$

If n is odd, for (E),

$$q(x) = \frac{axF_{2n+1}(x) + L_{2n+1}(x)}{2b(-F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{axF_{n+1}(x)L_n(x) + (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{-2bF_{n+1}(x)L_n(x)}$$
$$\overset{(13)}{=} -\frac{L_{n+1}(x)}{bL_n(x)}.$$

If n is even, for (F),

$$q(x) = \frac{axF_{2n+1}(x) - L_{2n+1}(x)}{2b(-F_{2n+1}(x) + b^n)} \overset{(7)(9)}{=} \frac{axL_{n+1}(x)F_n(x) - (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{-2bL_{n+1}(x)F_n(x)}$$

$$\overset{(14)}{=} \frac{L_n(x)}{L_{n+1}(x)}.$$

If n is odd, for (F),

$$q(x) = \frac{axF_{2n+1}(x) - L_{2n+1}(x)}{2b(-F_{2n+1}(x) + b^n)} \overset{(7)(9)(10)}{=} \frac{axL_{n+1}(x)F_n(x) - (a^2x^2 + 4b)F_{n+1}(x)F_n(x)}{-2bF_{n+1}(x)L_n(x)}$$

$$\overset{(14)}{=} \frac{F_n(x)}{F_{n+1}(x)}.$$

3.3. **Proof of Theorem 1.4.** In the proof of Theorem 1.2, we apply Proposition 2.3 instead of applying Proposition 2.1 and set $x = 1$ to complete the proof.

3.4. **Proof of Theorem 1.5.** In the proof of Theorem 1.3, we apply Proposition 2.3 instead of applying Proposition 2.1 and set $x = 1$ to complete the proof.

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National Institute of Technology, Wakayama College, 77 Noshima, Nada-cho, Gobo, Wakayama, Japan 644-0023
Email address: tsuno@wakayama-nct.ac.jp