KNIGHTS ARE 24/13 TIMES FASTER THAN THE KING

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ABSTRACT. On an infinite chess board, how much faster can the knight reach a square compared with the king, on average? More generally, for coprime $b > a \in \mathbb{Z}_{\geq 1}$ such that $a+b$ is odd, define the (a,b)-knight and the king as

$$\mathrm{N}_{a,b} = \{(a,b),(b,a),(-a,b),(-b,a),(-b,-a),(-a,-b),(a,-b),(b,-a)\},$$
$$\mathrm{K} = \{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\} \subseteq \mathbb{Z}^2,$$

respectively. One way to formulate this question is by asking for the average ratio, for $\mathbf{p} \in \mathbb{Z}^2$ in a box, between $\min\{h \in \mathbb{Z}_{\geq 1} \mid \mathbf{p} \in h\mathrm{N}\}$ and $\min\{h \in \mathbb{Z}_{\geq 1} \mid \mathbf{p} \in h\mathrm{K}\}$, where $hA = \{\mathbf{a}_1 + \cdots + \mathbf{a}_h \mid \mathbf{a}_1, \ldots, \mathbf{a}_h \in A\}$ is the h-fold sumset of A. We show that this ratio equals $2(a+b)b^2/(a^2+3b^2)$.

1. INTRODUCTION

Let $A \subseteq \mathbb{Z}^2$ be a finite set. For each $\mathbf{p} \in \mathbb{Z}^2$, we are interested in determining the smallest $h \geq 1$ for which we can write $\mathbf{p} = \mathbf{a}_1 + \cdots + \mathbf{a}_h$, where $\mathbf{a}_i \in A$ for $1 \leq i \leq h$ is not necessarily distinct. Writing $hA = \{\mathbf{a}_1 + \cdots + \mathbf{a}_h \mid \mathbf{a}_1, \ldots, \mathbf{a}_h \in A\}$ for the h-fold sumset of A, we define $A(0,0) := 0$ and, for $(x,y) \neq (0,0)$,

$$A(x,y) := \min\{h \geq 1 \mid (x,y) \in hA\}. \tag{1.1}$$

The study of the *size* of hA goes back to Khovanskii [3], who showed that $|hA|$ is given by a polynomial in terms of h for h sufficiently large (cf. Nathanson–Ruzsa [4] for a more combinatorial proof). In another direction, Granville–Shakan–Walker [1, 2] studied the *structure* of hA, showing that, roughly speaking, for every large h, every element that "could be" in hA is in hA.

In this note, we will study the behavior of $A(x,y)$ for a particular class of sumsets. Thinking of $\mathbb{Z}^2$ as an infinite chess board, a finite set A may be thought of as a *piece* placed at the origin, being able to move only to $\mathbf{a} \in A$. Then, in two moves, the piece is able to reach every point in $2A$, and so on. We say that A is

- *Primitive*, if $A(x,y)$ is well-defined for every $x, y \in \mathbb{Z}$;
- *Symmetric*, if $(a,b) \in A$ implies $(\delta_1 a, \delta_2 b), (\delta_1 b, \delta_2 a) \in A$ for every choice of $\delta_1, \delta_2 \in \{-1,+1\}$.

Notation: For real functions $f, g : \mathbb{R}_{>0} \to \mathbb{R}$, we write $f(x) = O(g(x))$ if there is an $M > 0$ such that $f(x) \leq Mg(x)$ for every large x.

1.1. **The King and the (a,b)-knight.** The two pieces that will concern us in this note are the following pieces.

a) The *king* $\mathrm{K} = \{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}$ is the smallest symmetric piece with $(1,0),(1,1) \in \mathrm{K}$. (see Figure 1)

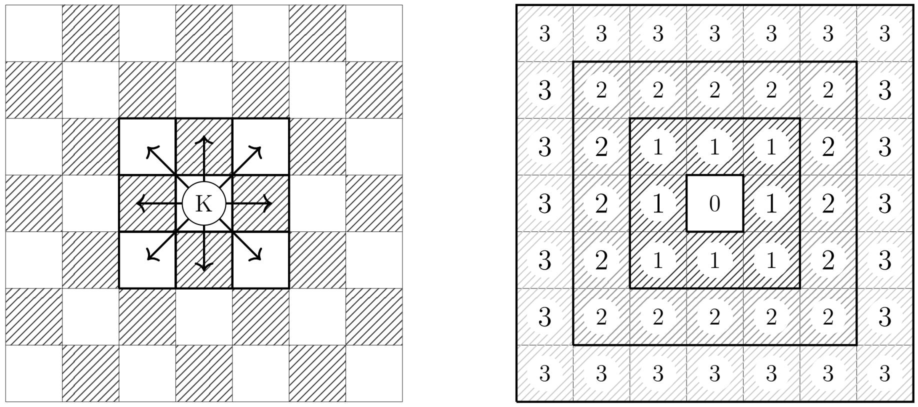

FIGURE 1. The king's movements (left) and $\mathrm{K}(x, y)$ (right).

b) For $a, b \in \mathbb{Z}_{\geq 1}$, we define the (a, b)-*knight* $\mathrm{N}_{a,b}$ by the set of moves

$$\mathrm{N}_{a,b} := \{\, (b, a),\ (a, b),\ (-a, b),\ (-b, a),\ (-b, -a),\ (-a, -b),\ (a, -b),\ (b, -a) \,\};$$

in other words, $\mathrm{N}_{a,b}$ is the smallest symmetric piece with $(a, b) \in \mathrm{N}_{a,b}$. The usual chess knight is the $(1, 2)$-knight, which we call just *knight* and denote it by N. (see Figure 2)

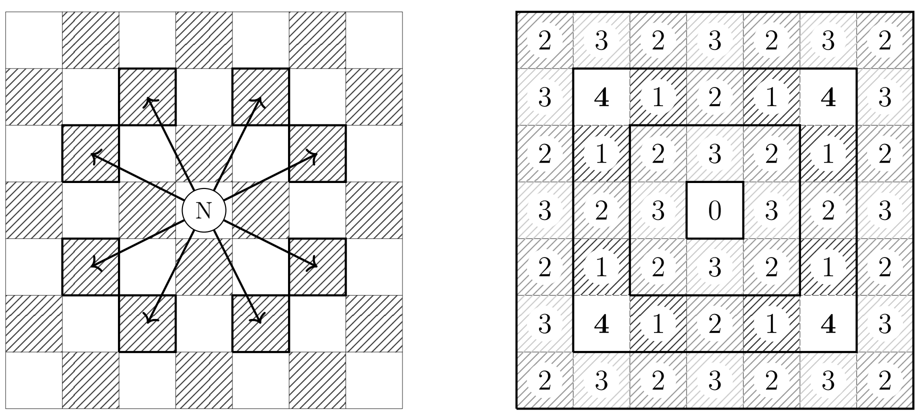

FIGURE 2. The knight's movements (left) and $\mathrm{N}(x, y)$ (right).

Not all (a, b)-knights are primitive. For $\mathrm{N}_{a,b}$ to be primitive, it is necessary and sufficient that $\gcd(a, b) = 1$ and $a + b$ be odd. To see this, color $\mathbb{Z}^2$ like a chess board (i.e., paint (x, y) white if $2 \mid x + y$, and black otherwise). The necessary direction is then easy: $\gcd(a, b) \mid \gcd(x, y)$ for every point (x, y) accessible to $\mathrm{N}_{a,b}$, and if $a + b$ is even, then $\mathrm{N}_{a,b}$ never accesses black points. For the sufficient direction, note that since $\mathrm{N}_{a,b}$ changes colors every move, it suffices to show that it can access all the white points, and by symmetry, it suffices to show that it accesses $(2, 0)$. Since $(b, a) + (b, -a) = (2b, 0)$ and $(a, b) + (a, -b) = (2a, 0)$, the (a, b)-knight can access every point of the form $(2(ax + by), 0)$ for $x, y \in \mathbb{Z}$; which, since $\gcd(a, b) = 1$, implies that $\mathrm{N}_{a,b}$ can access $(2, 0)$.

By the symmetries of $\mathrm{N}_{a,b}$, to understand the behavior of $\mathrm{N}_{a,b}(x,y)$, it suffices to study $x \geq y \in \mathbb{Z}_{\geq 0}$, where $\mathrm{K}(x,y) = x$. We will show the following theorem.

Theorem 1.1. *Let $b > a \geq 1$ be integers with $\gcd(a,b) = 1$ and $a+b$ odd, and let $x \geq y \in \mathbb{Z}_{\geq 0}$.*

(i) *If $y \leq \frac{a}{b}x$, then $\mathrm{N}_{a,b}(x,y) = \frac{x}{b} + O(b)$.*

(ii) *If $y > \frac{a}{b}x$, then $\mathrm{N}_{a,b}(x,y) = \frac{x+y}{a+b} + O(b)$.*

In Subsection 2.2, we describe the distribution of N/K.

1.2. **Average Velocity in a Box.** Each finite set A induces a metric $d_A(\mathbf{p},\mathbf{q}) := A(\mathbf{q}-\mathbf{p})$ for $\mathbf{p},\mathbf{q} \in \mathbb{Z}^2$. The king's metric coincides with the one induced by the max norm

$$\|(x,y)\|_\infty = \max\{|x|,|y|\},$$

and thus, we equip $\mathbb{Z}^2$ with this metric. For $h \geq 1$, write

$$\mathcal{B}_h := \{\mathbf{p} \in \mathbb{Z}^2 \mid \|\mathbf{p}\|_\infty \leq h\}, \qquad \mathcal{B}_h^* := \mathcal{B}_h \setminus \{(0,0)\}$$

for the ball and punctured ball of radius h, respectively. Note that $\mathcal{B}_h = \bigcup_{\ell=1}^{h} \ell\mathrm{K}$ and $\partial\mathcal{B}_h = \{\mathbf{p} \in \mathbb{Z}^2 \mid \|\mathbf{p}\|_\infty = h\} = h\mathrm{K} \setminus \bigcup_{\ell=0}^{\ell-1} \ell\mathrm{K}$. We have $|\partial\mathcal{B}_h| = 8h$ and $|\mathcal{B}_h^*| = 4h(h+1)$.

What is the average value of $A(x,y)$ in $\mathcal{B}_h$? For instance, the king K is such that $\mathrm{K}(x,y) = \ell$ if and only if $(x,y) \in \partial\mathcal{B}_\ell$; hence,

$$\frac{1}{|\mathcal{B}_h^*|} \sum_{\mathbf{p} \in \mathcal{B}_h^*} \mathrm{K}(\mathbf{p}) = \frac{1}{4h(h+1)} \sum_{\ell=1}^{h} \ell \cdot 8\ell = \frac{2h}{3} + \frac{1}{3}.$$

Thus, we consider the following notion of velocity, which can be understood intuitively as how fast the king K sees the piece A moving (see Remark 3.2).

Definition 1.2 (Velocity)**.** *For a finite primitive set $A \subseteq \mathbb{Z}^2$, the* average velocity $v = v_{\mathrm{K}}$ *of A is given by*

$$v(A) := \lim_{h \to +\infty} \frac{2h}{3} \left(\frac{1}{|\mathcal{B}_h|} \sum_{\mathbf{p} \in \mathcal{B}_h} A(\mathbf{p}) \right)^{-1}.$$

The number $v(A)$ may be thought of as controlling how fast A spreads through $\mathcal{B}_h$. Intuitively, from Theorem 1.1, one might conclude that the knight is almost, although not quite, *twice* as fast as the king. Points of the type $(x,0)$, for example, can be accessed by the knight in around $x/2$ moves, whereas points of the form (x,x) can be accessed in around $2x/3$ moves. We will show that

$$\frac{\sum_{\mathbf{p} \in \mathcal{B}_h} \mathrm{K}(\mathbf{p})}{\sum_{\mathbf{p} \in \mathcal{B}_h} \mathrm{N}(\mathbf{p})} \xrightarrow{h \to +\infty} \frac{24}{13};$$

in other words, the "not quite" is quantified by $2/13$. More generally, we have the following theorem.

Theorem 1.3. *Let $b > a \geq 1$ be integers with $\gcd(a,b) = 1$ and $a+b$ odd. Then*

$$v(\mathrm{N}_{a,b}) = \frac{2(a+b)b^2}{a^2+3b^2}.$$

See Remark 3.3 for a consequence of Theorem 1.3 when one takes a, b to be consecutive Fibonacci numbers — called *Fiboknights.*

2. Knights in $\mathbb{Z}^2$

We start with a lemma estimating how long the (a,b)-knight takes to access a point in $\mathcal{B}_{a+b}$.

Lemma 2.1. *Let $b > a \geq 1$ be integers with $\gcd(a,b) = 1$ and $a+b$ odd. For every $(x,y) \in \mathcal{B}_{a+b}$, we have $\mathrm{N}_{a,b}(x,y) = O(b)$ uniformly for a,b.*

Proof. Since $\gcd(a,b) = 1$, for every $1 \leq k \leq b$, there are $x,y \in \mathbb{Z}$ with $ax + by = k$, and we can select x,y such that $|x| \leq b$, $|y| \leq a$. Hence, since $\mathrm{N}_{a,b}$ is symmetric,

$$(2k,0) = x\big((a,b) + (a,-b)\big) + y\big((b,a) + (b,-a)\big)$$

is accessible in $2(|x| + |y|) \leq 2(a+b) < 4b$ moves, and so are the points $(-2k,0)$, $(0,2k)$, $(0,-2k)$. This implies that every point in $\mathcal{B}_{a+b}$ with even coordinates is accessible in $O(b)$ moves. By symmetry, it then suffices to show $\mathrm{N}_{a,b}(1,0) = O(b)$.

Suppose that a is even (so b is odd). Then, the point $(1-a,-b) \in \mathcal{B}_{a+b}$ has even coordinates, and so is accessible in $O(b)$ moves. Therefore, so is $(1,0) = (1-a,-b) + (a,b)$. The case when a is odd (so b is even) is similar. □

2.1. Proof of Theorem 1.1.

We prove the parts separately.

• Part (i): Let $\ell := \lfloor x/b \rfloor$, so that $\ell b \leq x < (\ell+1)b$ and $0 \leq y < (\ell+1)a$. Because $x \geq \ell b$, we have $\mathrm{N}_{a,b}(x,y) \geq \ell$. On the other hand, for each integer $0 \leq k \leq \ell/2$,

$$\big(\ell - k\big)(b,a) + k(b,-a) = (\ell b, (\ell - 2k)a),$$

so all the points in $\mathcal{S}_{(\ell b,\ell a)} := \{(\ell b, (\ell - 2k)a) \mid 0 \leq k \leq \ell/2\}$ are accessible in ℓ moves or less. All the points (x,y) with $\ell b \leq x < (\ell+1)b$ and $0 \leq y < (\ell+1)a$ are at a distance[1] at most $a+b$ from $\mathcal{S}_{(\ell b,\ell a)}$. By Lemma 2.1, $\mathrm{N}_{a,b}$ accesses all the points of $\mathcal{B}_{a+b}$ in $O(b)$ moves; it follows that $\mathrm{N}_{a,b}(x,y) \leq \ell + O(b)$.

• Part (ii): Let $t,u \in \mathbb{R}_{\geq 0}$ be such that $(x,y) = t(a,b) + u(b,a)$, so that $\mathrm{N}_{a,b}(x,y) \geq t + u$. Since

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff \frac{1}{b^2 - a^2} \begin{pmatrix} -a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ u \end{pmatrix},$$

we have $t = (by - ax)/(b^2 - a^2)$, $u = (bx - ay)/(b^2 - a^2)$ (both strictly positive, because $y/x > a/b$), and hence,

$$\mathrm{N}_{a,b}(x,y) \geq \frac{(b-a)(x+y)}{b^2 - a^2} = \frac{x+y}{a+b}.$$

On the other hand, $\lfloor t \rfloor (a,b) + \lfloor u \rfloor (b,a) = (x,y) + \mathbf{r}$, where $\mathbf{r} \in \mathcal{B}_{a+b}$. By Lemma 2.1, $\mathrm{N}_{a,b}$ accesses all the points of $\mathcal{B}_{a+b}$ in $O(b)$ moves; it follows that $\mathrm{N}_{a,b}(x,y) \leq \lfloor t \rfloor + \lfloor u \rfloor + O(b) = \frac{x+y}{a+b} + O(b)$. □

2.2. Distribution of N/K.

It follows from Theorem 1.1 that, for $x \geq y \in \mathbb{Z}_{\geq 0}$, the ratio $\frac{\mathrm{N}_{a,b}(x,y)}{\mathrm{K}(x,y)}$ lies essentially in between $\frac{1}{b}$ and $\frac{2}{a+b}$:

$$\frac{\mathrm{N}_{a,b}(x,y)}{\mathrm{K}(x,y)} = \begin{cases} \dfrac{1}{b} + O\!\left(\dfrac{b}{x}\right), & \text{if } \dfrac{y}{x} \leq \dfrac{a}{b}; \\[2ex] \dfrac{1}{a+b}\left(1 + \dfrac{y}{x}\right) + O\!\left(\dfrac{b}{x}\right), & \text{if } \dfrac{y}{x} > \dfrac{a}{b}. \end{cases}$$

[1] with respect to the max norm.

Analyzing this ratio in the box $\mathcal{B}_h$, one can study the *distribution* of $\mathrm{N}_{a,b}/\mathrm{K}$ via the real function

$$D_{a,b}(t) := \lim_{h\to+\infty} \frac{\#\{(x,y)\in\mathcal{B}_h \mid \frac{\mathrm{N}_{a,b}(x,y)}{\mathrm{K}(x,y)} \le t\}}{|\mathcal{B}_h|}.$$

The sets $\mathrm{N}_{a,b}$ and K are symmetric. Therefore, since $\frac{1}{a+b}(1+\frac{y}{x}) \le t$ if and only if $\frac{y}{x} \le (a+b)t-1$, and the proportion of points in $\mathcal{B}_h \cap \{(x,y)\in\mathbb{Z}_{\ge 0} \mid x \ge y\}$ with $\frac{y}{x}\le u$ equals $\frac{2}{h(h+1)}\sum_{x=1}^{h}\sum_{y=1}^{\lfloor ux\rfloor} 1 = u + O(1/h)$, we have

$$D_{a,b}(t) = \begin{cases} 0, & \text{if } t < \frac{1}{b};\\ (a+b)t-1, & \text{if } \frac{1}{b}\le t\le \frac{2}{a+b};\\ 1, & \text{if } t > \frac{2}{a+b}.\end{cases} \tag{2.1}$$

2.3. **Proof of Theorem 1.3.** By the symmetries of $\mathrm{N}_{a,b}(x,y)$, we have

$$\lim_{h\to+\infty}\frac{3}{2h}\left(\frac{1}{|\mathcal{B}_h|}\sum_{\mathbf{p}\in\mathcal{B}_h}\mathrm{N}_{a,b}(\mathbf{p})\right) = \lim_{h\to+\infty}\frac{3}{2h}\left(\frac{2}{h(h+1)}\sum_{\substack{x,y\in\mathbb{Z}_{\ge 0}\\ 1\le y\le x\le h}}\mathrm{N}_{a,b}(x,y)\right), \tag{2.2}$$

so it suffices to prove the existence and calculate the right side.

By Theorem 1.1, we have

$$\begin{aligned}\sum_{\substack{x,y\in\mathbb{Z}_{\ge 0}\\ 1\le y\le x\le h}}\mathrm{N}_{a,b}(x,y) &= \sum_{x=1}^{h}\left\lfloor\frac{a}{b}x\right\rfloor\frac{x}{b} + \sum_{x=1}^{h}\sum_{\substack{y=1\\ y/x>a/b}}^{x}\frac{x+y}{a+b} + O(bh^2)\\ &= \sum_{x=1}^{h}\left(\frac{a}{b^2} + \frac{1}{a+b}\sum_{\substack{y=1\\ y/x>a/b}}^{x}\left(1+\frac{y}{x}\right)\frac{1}{x}\right)x^2 + O(bh^2).\end{aligned}$$

Since

$$\begin{aligned}\sum_{\substack{y=1\\ y/x>a/b}}^{x}\left(1+\frac{y}{x}\right)\frac{1}{x} &= \frac{1}{x}\left(\sum_{\substack{y=1\\ y/x>a/b}}^{x}1\right) + \frac{1}{x^2}\left(\sum_{\substack{y=1\\ y/x>a/b}}^{x}y\right)\\ &= \left(1-\frac{a}{b}\right) + \frac{1}{2}\left(1-\frac{a^2}{b^2}\right) + O\left(\frac{1}{x}\right),\end{aligned}$$

it follows that

$$\sum_{\substack{x,y\in\mathbb{Z}_{\ge 0}\\ 1\le y\le x\le h}}\mathrm{N}_{a,b}(x,y) = \left(\frac{a}{b^2} + \frac{1}{a+b}\left(\left(1-\frac{a}{b}\right)+\frac{1}{2}\left(1-\frac{a^2}{b^2}\right)\right)\right)\frac{h(h+1)(2h+1)}{6} + O(bh^2).$$

Plugging this into the limit $v(\mathrm{N}_{a,b})$, we obtain

$$v(\mathrm{N}_{a,b}) = \lim_{h\to+\infty}\frac{2h}{3}\left(\frac{2}{h(h+1)}\sum_{\substack{x,y\in\mathbb{Z}_{\ge 0}\\ 1\le y\le x\le h}}\mathrm{N}_{a,b}(x,y)\right)^{-1}$$

$$= \left(\frac{a}{b^2} + \frac{1}{a+b}\left(\left(1-\frac{a}{b}\right) + \frac{1}{2}\left(1 - \frac{a^2}{b^2}\right)\right)\right)^{-1}$$
$$= \frac{2}{3}\left(\frac{2a^2+2ab}{3b^2} + \left(1-\frac{a}{b}\right)\left(1+\frac{a}{3b}\right)\right)^{-1}(a+b)$$
$$= \frac{2}{3}\left(1+\frac{1}{3}\frac{a^2}{b^2}\right)^{-1}(a+b) = \frac{2(a+b)b^2}{a^2+3b^2},$$

concluding the proof. □

3. Remarks

Remark 3.1. One checks that calculating the average using (2.1) agrees with (the inverse of) Theorem 1.3:

$$\mathbb{E}\left(\frac{\mathrm{N}_{a,b}}{\mathrm{K}}\right) := \int_0^{+\infty} (1 - D_{a,b}(t))\,\mathrm{d}t = \frac{1}{b} + \int_{1/b}^{2/(a+b)} (2-(a+b)t)\,\mathrm{d}t$$
$$= \frac{1}{b} + \frac{2(b-a)}{(b+a)b} - \frac{(b-a)(a+3b)}{2(a+b)b^2} = \frac{a^2+3b^2}{2(a+b)b^2}.$$

Remark 3.2 (On generality). The choice of the box $\mathcal{B}_h$ in Definition 1.2 is not generic, and different expanding regimes will give different answers for the ratio. In general, let $d \geq 2$ and $A \subseteq \mathbb{Z}^2$ be primitive set, and suppose that the origin $\mathbf{0}$ lies inside the convex hull $\mathcal{H}(A)$ of A. Write $A_{\mathbf{0}} = A \cup \{\mathbf{0}\}$. By Khovanskii's theorem [3, Corollary 1], we have $|hA_{\mathbf{0}}| = \mathrm{vol}(\mathcal{H}(A))\, h^d + O(h^{d-1})$ and

$$|hA_{\mathbf{0}} \setminus (h-1)A_{\mathbf{0}}| = d\,\mathrm{vol}(\mathcal{H}(A))\, h^{d-1} + O(h^{d-2}),$$

where $\mathrm{vol}(\mathcal{H}(A))$ denotes the d-volume of the convex hull of A. Thus,

$$\frac{1}{|hA_{\mathbf{0}}|}\sum_{\mathbf{p}\in hA_{\mathbf{0}}} A(\mathbf{p}) = \frac{1}{|hA_{\mathbf{0}}|}\sum_{\ell=1}^{h}\sum_{\mathbf{p}\in \ell A_{\mathbf{0}}\setminus(\ell-1)A_{\mathbf{0}}} A(\mathbf{p}) = \frac{dh}{d+1} + O(1).$$

Given a finite primitive set $B \subseteq \mathbb{Z}^d$, we define the velocity of B *relative to* A as

$$v_A(B) := \lim_{h\to+\infty}\left(1+\frac{1}{d}\right)h\left(\frac{1}{|hA_{\mathbf{0}}|}\sum_{\mathbf{p}\in hA_{\mathbf{0}}} B(\mathbf{p})\right)^{-1}.$$

It would be interesting to calculate the velocity of generalized knights with respect to the generalized king $\mathrm{K}^d = \{\mathbf{p} \in \mathbb{Z}^d \mid \|p\|_\infty = 1\}$, or velocities with respect to other pieces such as the *taxicab* $\mathrm{T} := \{\mathbf{p} = (x, y) \in \mathbb{Z}^2 \mid \|\mathbf{p}\|_1 := |x| + |y| = 1\} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$.

Remark 3.3 (Fiboknights). Fibonacci numbers $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$) satisfy the property that F_{3n} is even, F_{3n+1}, F_{3n+2} are odd, and $\gcd(F_n, F_{n+1}) = 1$. Define the *nth Fiboknight* as

$$\mathrm{FN}_n = \mathrm{N}_{F_{n+1},F_{n+2}},$$

so that the usual knight is the first Fiboknight. By the properties of Fibonacci numbers, FN_n is only primitive for n such that $3 \nmid n$.

Let $k \geq 1$, and let $n \to \infty$ through $n \in \mathbb{Z}_{\geq 1}$ for which FN_n, FN_{n+k} are primitive. Then, by Theorem 1.3, writing $\phi = \frac{1+\sqrt{5}}{2}$ for the golden ratio, we have

$$\begin{aligned}\lim_{\substack{n\to\infty\\ 3\nmid n,\, n+k}} \frac{v(\mathrm{FN}_{n+k})}{v(\mathrm{FN}_n)} &= \lim_{\substack{n\to\infty\\ 3\nmid n,\, n+k}} \frac{\dfrac{2(F_{n+k+1}+F_{n+k+2})F_{n+k+2}^2}{F_{n+k+1}^2+3F_{n+k+2}^2}}{\dfrac{2(F_{n+1}+F_{n+2})F_{n+2}^2}{F_{n+1}^2+3F_{n+2}^2}} \\ &= \lim_{\substack{n\to\infty\\ 3\nmid n,\, n+k}} \frac{F_{n+k+3}}{F_{n+3}} \frac{F_{n+k+2}^2}{F_{n+2}^2} \frac{(F_{n+1}^2+3F_{n+2}^2)}{(F_{n+k+1}^2+3F_{n+k+2}^2)} \\ &= \phi^k\,\phi^{2k}\,\frac{1+3\phi^2}{\phi^{2k}+3\phi^{2k+2}} \\ &= \phi^k.\end{aligned}$$

In particular, the ratio of the velocity of consecutive Fiboknights (which can only be of the form FN_{3n+1}, FN_{3n+2}) converges to ϕ. In general, for fixed $m, k \geq 1$,

$$\lim_{\substack{n\to\infty\\ \text{primitive}}} \frac{v(\mathrm{N}_{F_{n+k},F_{n+m+k}})}{v(\mathrm{N}_{F_n,F_{n+m}})} = \frac{\dfrac{2(\phi^k+\phi^{m+k})\phi^{2(m+k)}}{\phi^{2k}+3\phi^{2(m+k)}}}{\dfrac{2(1+\phi^m)\phi^{2m}}{1+3\phi^{2m}}} = \phi^k.$$

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References

[1] A. Granville and G. Shakan, *The Frobenius postage stamp problem, and beyond*, Acta Math. Hungarica, **161.2** (2020), 700–718.

[2] A. Granville, G. Shakan, and A. Walker, *Effective results on the size and structure of sumsets*, Combinatorica, **43** (2023), 1139–1178.

[3] A. Khovanskii, *The Newton polytope, the Hilbert polynomial and sums of finite sets*, Funktsional. Anal. i Prilozhen, **26.4** (1992), 57–63, 96.

[4] M. B. Nathanson and I. Z. Ruzsa, *Polynomial growth of sumsets in abelian semigroups*, J. Théor. Nombr. Bordx., **14.2** (2002), 553–560.

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