WEIGHTED SCHREIER-TYPE SETS AND THE FIBONACCI SEQUENCE

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Abstract. For a finite set $A \subset \mathbb{N}$ and $k \in \mathbb{N}$, let $\omega_k(A) = \sum_{i\in A, i\neq k} 1$. For each $n \in \mathbb{N}$, define

$$a_{k,n} \ = \ |\{E \subset \mathbb{N} \, : \, E = \emptyset \text{ or } \omega_k(E) < \min E \leqslant \max E \leqslant n\}|.$$

We prove that

$$a_{k,k+\ell} \ = \ 2F_{k+\ell} \text{ for all } \ell \geqslant 0 \text{ and } k \geqslant \ell + 2,$$

where F_n is the nth Fibonacci number.

1. Introduction

Recall that a finite set $E \subset \mathbb{N}$ is called a *Schreier* set if $\min E \geqslant |E|$. Denote the collection of all Schreier sets by $\mathcal{S}$, which includes the empty set. Sets E with $\min E > |E|$ ($\min E = |E|$, respectively) are called *nonmaximal Schreier* sets (*maximal Schreier* sets, respectively). The empty set is vacuously maximal and nonmaximal. We let S^{NMAX} consist of nonmaximal Schreier sets and let S^{MAX} consist of maximal Schreier sets. Bird [4] showed that for each positive integer n,

$$|\{E \in \mathcal{S} \, : \, \max E = n\}| \ = \ F_n,$$

where $(F_n)_{n=0}^{\infty}$ is the Fibonacci sequence defined as: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. Beanland, et al. [2] generalized the Schreier condition to $q \min E \geqslant p|E|$ and established an inclusion-exclusion type recurrence for the sequence

$$m_{p,q,n} \ := \ |\{E \subset \mathbb{N} \, : \, q \min E \geqslant p|E| \text{ and } \max E = n\}|. \tag{1.1}$$

In particular, [2, Theorem 1] gives

$$m_{p,q,n} \ = \ \sum_{k=1}^{q} (-1)^{k+1} \binom{q}{k} m_{p,q,n-k} + m_{p,q,n-(p+q)} \text{ for } n \geqslant p + q.$$

Recently, Beanland, et al. [3] studied unions of Schreier sets and proved a linear recurrence for their counts using recursively defined characteristic polynomials. Relations between Schreier sets and partitions, compositions, and Turán graphs have also been discovered [1, 7, 8].

Previous work gave a uniform weight to each number when measuring the size of a set. For example, rewriting $q \min E \geqslant p|E|$ in (1.1) as $\min E \geqslant (p/q)|E|$, one can think of $(p/q)|E|$ as the size of E, where each element is given the same weight p/q. In this paper, we shall assign the weight 0 to one number and the weight 1 to the other numbers. Specifically, let $\mathcal{N}$ be the collection of finite subsets of $\mathbb{N}$. For $E \in \mathcal{N}$ and $k \in \mathbb{N}$, define $\omega_k(E) := \sum_{n\in E, n\neq k} 1$ and

$$\mathcal{S}^{(k)} \ := \ \{E \in \mathcal{N} \, : \, E = \emptyset \text{ or } \min E > \omega_k(E)\}.$$

For example, $\mathcal{S}^{(1)} = \mathcal{S}^{NMAX} \cup \{1\}$, $\mathcal{S}^{(2)} = \mathcal{S}^{NMAX} \cup \{\{2, n\} : 2 < n\}$, and $\mathcal{S}^{(3)} = \mathcal{S}^{NMAX} \cup \{\{2, 3\}\} \cup \{\{3, n_1, n_2\} : 3 < n_1 < n_2\}$. Generally, we have the following equality, whose proof is in the appendix:

$$\mathcal{S}^{(k)} \ = \ \mathcal{S}^{NMAX} \cup \{E \in \mathcal{S}^{MAX} \, : \, k \in E\}. \tag{1.2}$$

The work was partially supported by the College of Arts & Sciences at Texas A&M University.

For $n, k \in \mathbb{N}$, let

$$\mathcal{A}_{k,n} \ := \ \{E \in \mathcal{S}^{(k)} \ : \ E = \emptyset \text{ or } \ \max E \leqslant n\} \text{ and } a_{k,n} = |\mathcal{A}_{k,n}|.$$

We can write

$$\mathcal{A}_{k,n} \ = \ \{E \in \mathcal{N} \ : \ E = \emptyset \text{ or } \omega_k(E) < \min E \leqslant \max E \leqslant n\}.$$

We record values of $a_{k,n}$ for small k and n in Table 1.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	6	9	14	22	35	56	90	145	234	378	611	988	1598
2	1	2	4	7	11	17	26	40	62	97	153	243	388	622	1000	1611
3	1	2	4	6	10	17	28	45	71	111	173	270	423	666	1054	1676
4	1	2	3	6	10	16	26	43	71	116	187	298	471	741	1164	1830
5	1	2	3	5	10	16	26	42	68	111	182	298	485	783	1254	1995
6	1	2	3	5	8	16	26	42	68	110	178	289	471	769	1254	2037
7	1	2	3	5	8	13	26	42	68	110	178	288	466	755	1226	1995

Table 1. Initial numbers $a_{k,n}$ for different k and n.

Observe that when $k > n$, the $a_{k,n}$s are Fibonacci numbers. This is expected because when $k > n$, $\omega_k(E) = |E|$ for all sets $E \subset \{1, \ldots, n\}$, so the weight ω_k gives the cardinality of sets as in the result by Bird [4]. When $k = n$, we witness the Fibonacci numbers multiplied by 2 (see Table 2). When $n > k$, we shall see that the $a_{k,n}$s are found by iterated partial sums.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2															
2		2														
3			4													
4				6												
5					10											
6						16										
7							26									

Table 2. The diagonal $a_{n,n}$.

The following confirms that the pattern in Table 2 holds for all n.

Theorem 1.1. *For $n \in \mathbb{N}$, it holds that*

$$a_{n,n} \ = \ 2F_n. \tag{1.3}$$

We shall use Theorem 1.1 to establish a general formula for $a_{k,n}$ for each pair $(k, n) \in \mathbb{N}^2$.

Theorem 1.2. *It holds that*

$$a_{k,k+\ell} \ = \ \begin{cases} F_{\ell+2} + 1, & \text{if } k = 1, \ell \geqslant 0; \\ 2\sum_{i=0}^{k-2} \binom{\ell}{i} F_{k-i} + 2\binom{\ell}{k-1} + \sum_{j=1}^{\ell} \binom{j}{\ell-j+k}, & \text{if } k \geqslant 2, \ell \geqslant 0; \\ F_{k+\ell+1}, & \text{if } k \geqslant 2, -k < \ell < 0. \end{cases}$$

Example 1.3. Let $k = 4$ and $\ell = 6$. Table 1 gives us $a_{4,10} = 116$, whereas Theorem 1.2 gives

$$\begin{aligned} & 2\sum_{i=0}^{2} \binom{6}{i} F_{4-i} + 2\binom{6}{3} + \sum_{j=1}^{6} \binom{j}{10-j} \\ & = \ 2\left(\binom{6}{0} F_4 + \binom{6}{1} F_3 + \binom{6}{2} F_2\right) + 2\binom{6}{3} + \binom{5}{5} + \binom{6}{4} \ = \ 116, \end{aligned}$$

as expected.

As a corollary of Theorem 1.2, we obtain a more general identity than the one in Theorem 1.1.

Theorem 1.4. *For $\ell \geqslant 0$ and $k \geqslant \ell + 2$, we have*

$$a_{k,k+\ell} \;=\; 2F_{k+\ell}.$$

Our paper is structured as follows: Section 2 proves Theorem 1.1, and Section 3 uses Theorem 1.1 and properties of iterated partial sums to prove Theorem 1.2.

2. The Sequence $(a_{n,n})_{n\geqslant 1}$

We shall present two proofs of Theorem 1.1. The first uses bijective maps, and the second uses the two well-known formulas:

$$\sum_{i=m}^{n}\binom{i}{m} \;=\; \binom{n+1}{m+1} \text{ for } n, m \in \mathbb{N}, \text{ and} \tag{2.1}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} \;=\; F_{n+1} \text{ for } n \geqslant 0. \tag{2.2}$$

Bijective proof of Theorem 1.1. Since $a_{1,1} = a_{2,2} = 2$, it suffices to show that $a_{n+1,n+1} = a_{n,n} + a_{n-1,n-1}$ for $n \geqslant 2$. Consider the following map $\Psi_1 : \mathcal{A}_{n-1,n-1} \to \mathcal{A}_{n+1,n+1}$ defined as

$$\Psi_1(F) \;=\; (F+1) \cup \{n+1\}.$$

Then Ψ_1 is well-defined because $\Psi_1(\emptyset) = \{n+1\} \in \mathcal{A}_{n+1,n+1}$, and if $F \neq \emptyset$,

$$\min \Psi_1(F) \;=\; \min F + 1 \;\geqslant\; |F| + 1 \;=\; |\Psi_1(F)| \;>\; |\Psi_1(F)| - 1 \;=\; \omega_{n+1}(\Psi_1(F)). \tag{2.3}$$

Furthermore, $\Psi_1(F)$ is clearly one-to-one.

Define the following map $\Psi_2 : \mathcal{A}_{n,n} \to \mathcal{A}_{n+1,n+1}\backslash\Psi_1(\mathcal{A}_{n-1,n-1})$ as

$$\Psi_2(F) \;=\; \begin{cases} F, & \text{if } F \in \mathcal{S}^{NMAX}; \\ (F\backslash\{n\}) \cup \{n+1\}, & \text{if } F \in \mathcal{S}^{MAX}\backslash\{\emptyset\}. \end{cases}$$

We show that Ψ_2 is well-defined. Obviously, $\Psi_2(\mathcal{A}_{n,n}) \subset \mathcal{A}_{n+1,n+1}$. Let us show that $\Psi_2(\mathcal{A}_{n,n}) \cap \Psi_1(\mathcal{A}_{n-1,n-1}) = \emptyset$. Pick $F \in \mathcal{A}_{n,n}$.

Case 1: If $F \in \mathcal{S}^{NMAX}$, then $\Psi_2(F) = F$ and $n+1 \notin \Psi_2(F)$, so $\Psi_2(F) \notin \Psi_1(\mathcal{A}_{n-1,n-1})$.

Case 2: If $F \in \mathcal{S}^{MAX}\backslash\{\emptyset\}$, then (1.2) implies that $n \in F$. That F contains $n \geqslant 2$ and $F \in \mathcal{S}^{MAX}$ imply that $|F| \geqslant 2$. Therefore,

$$|\Psi_2(F)| \;=\; |F| \;=\; \min F \;=\; \min \Psi_2(F). \tag{2.4}$$

Suppose, for a contradiction, that there exists $G \in \mathcal{A}_{n-1,n-1}$ such that $\Psi_1(G) = \Psi_2(F)$. By (2.4), $\min \Psi_1(G) = |\Psi_1(G)|$. It follows from (2.3) that $\min G = |G|$. By (1.2), we must have $n-1 \in G$. Hence, $n \in \Psi_1(G)$, contradicting that $n \notin \Psi_2(F)$. We have shown that $\Psi_2(\mathcal{A}_{n,n}) \cap \Psi_1(\mathcal{A}_{n-1,n-1}) = \emptyset$.

Furthermore, Ψ_2 is one-to-one. Indeed, suppose that $\Psi_2(F_1) = \Psi_2(F_2)$. If $n+1 \notin \Psi_2(F_1) = \Psi_2(F_2)$, then by the definition of Ψ_2, we have

$$F_1 \;=\; \Psi_2(F_1) \;=\; \Psi_2(F_2) \;=\; F_2.$$

If $n+1 \in \Psi_2(F_1) = \Psi_2(F_2)$, then $F_1, F_2 \in \mathcal{S}^{MAX}\backslash\{\emptyset\}$, so (1.2) implies that $n \in F_1 \cap F_2$. Therefore, $\Psi_2(F_1) = \Psi_2(F_2)$ guarantees that $F_1 = F_2$.

Finally, we show that Ψ_2 is surjective. Take a nonempty $F \in \mathcal{A}_{n+1,n+1}\backslash\Psi_1(\mathcal{A}_{n-1,n-1})$.

Case 1: $F \in \mathcal{S}^{NMAX}$. Then $n+1 \notin F$ because otherwise, $\Psi_1(F\backslash\{n+1\}-1) = F$. By (1.2), $F \in \mathcal{S}^{NMAX}$, so $\Psi_2(F) = F$.

Case 2: $F \in \mathcal{S}^{MAX}$. Then $n+1 \in F$. Let $E = (F\backslash\{n+1\}) \cup \{n\}$. It follows that $|F| \geqslant |E|$. Note that $F \notin \Psi_1(\mathcal{A}_{n-1,n-1})$ implies that $F \neq \{n+1\}$. Hence, $|F| \geqslant 2$ and so,

$$\min E \;=\; \min F \;>\; |F|-1 \;\geqslant\; |E|-1 \;=\; \omega_n(E).$$

Therefore, $E \in \mathcal{A}_{n,n}$. To show that $\Psi_2(E) = F$, it remains to verify that $n \notin F$. Suppose otherwise, i.e., $n \in F$. Let $H = F\backslash\{n+1\} - 1$. We have

$$\min H \;=\; \min F - 1 \;>\; \omega_{n+1}(F) - 1 \;=\; |F| - 2 \;=\; |H| - 1 \;=\; \omega_{n-1}(H).$$

Hence, $F = \Psi_1(H) \in \Psi(\mathcal{A}_{n-1,n-1})$, a contradiction. This completes our proof. □

Alternative proof of Theorem 1.1. For $n \in \mathbb{N}$, we build a set $A \subset \{1, \ldots, n\}$ with $\min A > \omega_n(A)$.

If $n \notin A$, then $\omega_n(A) = |A|$. Set $k = \min A \in \{1, \ldots, n-1\}$. For each such k, the set $A\backslash\{k\} \subset \{k+1, k+2, \ldots, n-1\}$. Since $k > |A|$, $|A\backslash\{k\}| \leqslant k-2$. Hence, the number of sets $A \in \mathcal{A}_{n,n}$ with $n \notin A$ is

$$c = \sum_{k=1}^{n-1}\sum_{j=0}^{k-2}\binom{n-k-1}{j}.$$

If $n \in A$, then $\omega_n(A) = |A| - 1$. Set $k = \min A \in \{1, \ldots, n\}$. For each $k \neq n$, the set $A\backslash\{k, n\}$ is a subset of $\{k+1, k+2, \ldots, n-1\}$ and has size

$$|A\backslash\{k,n\}| \;=\; |A| - 2 \;=\; (|A|-1) - 1 \;<\; k-1.$$

Hence, the number of sets $A \in \mathcal{A}_{n,n}$ with $n \in A$ and $A \neq \{n\}$ is also c.

If we include the empty set and $\{n\}$, we obtain

$$a_{n,n} \;=\; 2 + 2\sum_{k=1}^{n-1}\sum_{j=0}^{k-2}\binom{n-k-1}{j}.$$

It suffices to show that

$$\sum_{k=1}^{n-1}\sum_{j=0}^{k-2}\binom{n-k-1}{j} \;=\; F_n - 1.$$

Exchanging the order of the double sum and using (2.1) and (2.2), we have

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{j=0}^{k-2}\binom{n-k-1}{j} \;&=\; \sum_{j=0}^{n-3}\sum_{k=j+2}^{n-1}\binom{n-k-1}{j} \;=\; \sum_{j=0}^{\lfloor (n-3)/2\rfloor}\sum_{k=j}^{n-3-j}\binom{k}{j}\\ &=\; \sum_{j=0}^{\lfloor (n-3)/2\rfloor}\binom{n-2-j}{j+1} \;=\; \sum_{j=1}^{\lfloor (n-1)/2\rfloor}\binom{n-1-j}{j}\\ &=\; F_n - 1,\end{aligned}$$

as desired. □

3. The Table $(a_{k,n})_{k,n\geqslant 1}$

The goal of this section is to prove Theorem 1.2, giving formulas to compute $a_{k,n}$ for any $(k,n)\in\mathbb{N}^2$. We shall describe the proof idea for Theorem 1.2 after proving the next proposition and introducing the partial sum operators.

Proposition 3.1. *For $n>\max\{k,2\}$, we have*

$$a_{k,n} \;=\; a_{k,n-1}+a_{k-1,n-2}. \tag{3.1}$$

Proof. It is obvious from the definition of $\mathcal{A}_{k,n}$ that $\mathcal{A}_{k,n-1}\subset\mathcal{A}_{k,n}$. It suffices to show that there is a bijective map between $\mathcal{A}_{k,n}\backslash\mathcal{A}_{k,n-1}$ and $\mathcal{A}_{k-1,n-2}$. To do so, we define $\Psi:\mathcal{A}_{k-1,n-2}\to\mathcal{A}_{k,n}\backslash\mathcal{A}_{k,n-1}$ as

$$\Psi(F)\;:=\;(F+1)\cup\{n\}.$$

We check that Ψ is well-defined. Let $F\in\mathcal{A}_{k-1,n-2}$ and $E=\Psi(F)$. If $F=\emptyset$, then $E=\{n\}\in\mathcal{A}_{k,n}\backslash\mathcal{A}_{k,n-1}$. If $F\neq\emptyset$, we have $\max E=n$ and

$$\min E\;=\;\min F+1\;>\;\omega_{k-1}(F)+1\;=\;\omega_k(F+1)+1\;=\;\omega_k((F+1)\cup\{n\})\;=\;\omega_k(E).$$

Thus, Ψ is well-defined.

Clearly, Ψ is injective. Let us verify that Ψ is surjective. Pick $E\in\mathcal{A}_{k,n}\backslash\mathcal{A}_{k,n-1}$. Then $n\in E$. Let $F=E\backslash\{n\}-1$. We claim that $F\in\mathcal{A}_{k-1,n-2}$. Indeed, if $E=\{n\}$, then $F=\emptyset$. If $\{n\}\subsetneq E$, then

$$\max F\;=\;\max E\backslash\{n\}-1\;\leqslant\;n-1-1\;=\;n-2.$$

Furthermore,

$$\min F\;=\;\min E-1\;>\;\omega_k(E)-1\;=\;\omega_k((F+1)\cup\{n\})-1\;=\;\omega_k(F+1)\;=\;\omega_{k-1}(F).$$

This completes the proof. ☐

Next, we collect useful properties of iterated partial sums of a sequence. Given a sequence $(a_n)_{n=0}^\infty$ and a number b, the partial sum operator P_b produces the sequence $(a'_n)_{n=0}^\infty$ defined as follows:

$$a'_0\;=\;b,\quad a'_1\;=\;b+a_0,\quad a'_2\;=\;b+a_0+a_1,\quad\ldots.$$

In general,

$$P_b((a_n)_{n=0}^\infty)\;:=\;\left(a'_n:=b+\sum_{i=0}^{n-1}a_i\right)_{n\geqslant 0}.$$

Fix a sequence $\mathbf{s}=(b_n)_{n=0}^\infty$ and let $(t^{\mathbf{s}}_{k,n})_{n=0}^\infty$, for $k\geqslant 0$, denote the sequence

$$P_{b_{k-1}}(\cdots P_{b_1}(P_{b_0}((a_n)_{n=0}^\infty))).$$

Here, $(t^{\mathbf{s}}_{0,n})_{n=0}^\infty=(a_n)_{n=0}^\infty$. In the special case that $\mathbf{s}$ consists of only 0, the sequence $(t^{\mathbf{s}}_{k,n})_{n\geqslant 0}$ is what is known as *the k-partial sum of* $(a_n)_{n\geqslant 0}$, which shall be denoted by $P^{(k)}((a_n)_{n\geqslant 0})$, and its mth term is denoted by $P^{(k)}((a_n)_{n\geqslant 0})(m)$. As a result,

$$P^{(k)}((a_n)_{n\geqslant 0})(m)\;=\;t^{\mathbf{s}}_{k,m}\text{ for all }m\geqslant 0.$$

We are ready to describe the proof idea for Theorem 1.2. The most technical case is when $k\geqslant 2$ and $\ell\geqslant 0$. When $\ell=0$, Theorem 1.1 gives

$$a_{k,k+\ell}\;=\;a_{k,k}\;=\;2F_k.$$

Let us focus on when $\ell>0$. Proposition 3.1 states that

$$a_{k,k+\ell}\;=\;a_{k,k+\ell-1}+a_{k-1,k+\ell-2}.$$

Hence, $(a_{k,k+\ell})_{\ell\geqslant 0}$ is $P_{a_{k,k}}((a_{k-1,k-1+n})_{n\geqslant 0})$. For example, it is readily checked that $(a_{3,n})_{n\geqslant 3}$ is $P_{a_{3,3}}((a_{2,2+n})_{n\geqslant 0})$. Applying Proposition 3.1 repeatedly, we obtain that

$$\begin{aligned}(a_{k,k+\ell})_{\ell\geqslant 0} &= P_{a_{k,k}}P_{a_{k-1,k-1}}\cdots P_{a_{2,2}}((a_{1,1+n})_{n\geqslant 0})\\ &= P_{a_{k,k}}P_{a_{k-1,k-1}}\cdots P_{a_{2,2}}((F_{n+2}+1)_{n\geqslant 0}).\end{aligned}$$

Therefore, to compute $(a_{k,k+\ell})_{\ell\geqslant 0}$, we shall equivalently compute

$$P_{a_{k,k}}P_{a_{k-1,k-1}}\cdots P_{a_{2,2}}((F_{n+2}+1)_{n\geqslant 0}),$$

which is a slight modification of the partial sums of the Fibonacci numbers

$$\underbrace{P_0P_0\cdots P_0}_{k-1}((F_n)_{n\geqslant 0}) \;=:\; P^{(k-1)}((F_n)_{n\geqslant 0}).$$

But $P^{(k-1)}((F_n)_{n\geqslant 0})$ is well-known from [6, Theorem 1.3], which states that

$$P^{(k)}((F_n)_{n\geqslant 0})(\ell) \;=\; \sum_{j=0}^{\ell-1}\binom{j}{\ell-1-j+k} \text{ for } k,\ell\geqslant 0. \tag{3.2}$$

The above observations suggest that we should transform $P_{a_{k,k}}P_{a_{k-1,k-1}}\cdots P_{a_{2,2}}((F_{n+2}+1)_{n\geqslant 0})$ into $P^{(k-1)}((F_n)_{n\geqslant 0})$. The next few results serve this goal: Lemma 3.2 connects $P_{a_{k,k}}P_{a_{k-1,k-1}}\cdots P_{a_{2,2}}$ and $\underbrace{P_0P_0\cdots P_0}_{k-1} = P^{(k-1)}$, whereas Lemma 3.3 connects $P^{(k-1)}((F_{n+2}+1)_{n\geqslant 0})$ and $P^{(k-1)}((F_{n+2})_{n\geqslant 0})$.

Back to the notation introduced above, let $\mathbf{s}_1$ consist of only 0 and $\mathbf{s}_2 = (b_n)_{n\geqslant 0}$. Our following lemma compares, for each $k\in\mathbb{N}$, the two sequences $(t^{\mathbf{s}_1}_{k,n})_{n\geqslant 0}$ and $(t^{\mathbf{s}_2}_{k,n})_{n\geqslant 0}$. The proof is standard, using double induction.

Lemma 3.2. *Let $k\geqslant 0$ and let $\mathbf{s}_1$ consist of only 0 and $\mathbf{s}_2 = (b_n)_{n\geqslant 0}$. Then*

$$t^{\mathbf{s}_2}_{k,n} - t^{\mathbf{s}_1}_{k,n} \;=\; \sum_{i=0}^{k-1}\binom{n}{i} b_{k-1-i} \text{ for all } n\geqslant 0. \tag{3.3}$$

Proof. We prove by induction on k. For the base case, (3.3) clearly holds for $k\in\{0,1\}$. Suppose that (3.3) holds for all $k\leqslant j$ for some $j\geqslant 1$. We shall show that (3.3) holds for $k=j+1$. To do so, we proceed by induction on $n\geqslant 0$, given that $k=j+1$ and that (3.3) holds for all $k\leqslant j$. For the base case, by definitions, we have

$$t^{\mathbf{s}_2}_{j+1,0} - t^{\mathbf{s}_1}_{j+1,0} \;=\; b_j - 0 \;=\; b_j \;=\; \sum_{i=0}^{j}\binom{0}{i} b_{j-i};$$

hence, (3.3) is true for $k=j+1$ and $n=0$. Assume that (3.3) is true for $k=j+1$ and all $n\leqslant \ell$ for some $\ell\geqslant 0$. We show that (3.3) is true for $k=j+1$ and $n=\ell+1$. Indeed, by the two inductive hypotheses, we have

$$\begin{aligned}
t^{\mathbf{s}_2}_{j+1,\ell+1} - t^{\mathbf{s}_1}_{j+1,\ell+1} &= \left(t^{\mathbf{s}_2}_{j+1,\ell} + t^{\mathbf{s}_2}_{j,\ell}\right) - \left(t^{\mathbf{s}_1}_{j+1,\ell} + t^{\mathbf{s}_1}_{j,\ell}\right) \\
&= \left(t^{\mathbf{s}_2}_{j+1,\ell} - t^{\mathbf{s}_1}_{j+1,\ell}\right) + \left(t^{\mathbf{s}_2}_{j,\ell} - t^{\mathbf{s}_1}_{j,\ell}\right) \\
&= \sum_{i=0}^{j}\binom{\ell}{i} b_{j-i} + \sum_{i=0}^{j-1}\binom{\ell}{i} b_{j-1-i} \\
&= \sum_{i=0}^{j}\binom{\ell}{i} b_{j-i} + \sum_{i=1}^{j}\binom{\ell}{i-1} b_{j-i} \\
&= b_j + \sum_{i=1}^{j}\left(\binom{\ell}{i-1} + \binom{\ell}{i}\right) b_{j-i} \ = \ \sum_{i=0}^{j}\binom{\ell+1}{i} b_{j-i}.
\end{aligned}$$

This completes our proof. □

Lemma 3.3. *Fix $k \geqslant 0$ and a sequence $(a_n)_{n\geqslant 0}$. Then*

$$P^{(k)}((a_n+1)_{n\geqslant 0})(m) - P^{(k)}((a_n)_{n\geqslant 0})(m) \ = \ \binom{m}{k} \text{ for all } m \geqslant 0. \tag{3.4}$$

Proof. We prove by induction on k. Base case: (3.4) is clearly true when $k = 0$. Inductive hypothesis: assume that (3.4) holds for $k \leqslant \ell$ for some $\ell \geqslant 0$. We show that (3.4) holds for $k = \ell + 1$, i.e.,

$$P^{(\ell+1)}((a_n+1)_{n\geqslant 0})(m) - P^{(\ell+1)}((a_n)_{n\geqslant 0})(m) \ = \ \binom{m}{\ell+1} \text{ for all } m \geqslant 0. \tag{3.5}$$

To do so, we induct on m. For $m = 0$, both sides of (3.5) are equal to 0. Inductive hypothesis: suppose that (3.5) is true for all $m \leqslant s$ for some $s \geqslant 0$. We have

$$\begin{aligned}
&P^{(\ell+1)}((a_n+1)_{n\geqslant 0})(s+1) - P^{(\ell+1)}((a_n)_{n\geqslant 0})(s+1) \\
&= (P^{(\ell+1)}((a_n+1)_{n\geqslant 0})(s) + P^{(\ell)}((a_n+1)_{n\geqslant 0})(s)) \\
&\qquad - (P^{(\ell+1)}((a_n)_{n\geqslant 0})(s) + P^{(\ell)}((a_n)_{n\geqslant 0})(s)) \\
&= (P^{(\ell+1)}((a_n+1)_{n\geqslant 0})(s) - P^{(\ell+1)}((a_n)_{n\geqslant 0})(s)) \\
&\qquad + (P^{(\ell)}((a_n+1)_{n\geqslant 0})(s) - P^{(\ell)}((a_n)_{n\geqslant 0})(s)) \\
&= \binom{s}{\ell+1} + \binom{s}{\ell} \ = \ \binom{s+1}{\ell+1}.
\end{aligned}$$

This completes our proof. □

The proof of the following lemma is similar to those of Lemmas 3.2 and 3.3, so we move the proof to the appendix.

Lemma 3.4. *For all $k, \ell \geqslant 0$, we have*

$$P^{(k)}((F_{n+2})_{n\geqslant 0})(\ell) \ = \ P^{(k)}((F_n)_{n\geqslant 0})(\ell) + P^{(k)}((F_n)_{n\geqslant 0})(\ell+1). \tag{3.6}$$

The following easy observation shall be used in due course.

Proposition 3.5. *For $k > n \in \mathbb{N}$, we have*

$$a_{k,n} \ = \ F_{n+1} \text{ for } k > n.$$

Proof. For $k > n$, we have

$$\begin{aligned} a_{k,n} \;=\; |\mathcal{A}_{k,n}| \;&=\; |\{E \in \mathcal{S}^{(k)} \,:\, E = \emptyset \text{ or } \max E \leqslant n\}| \\ &=\; |\{E \subset \{1,2,\ldots,n\} \,:\, E = \emptyset \text{ or } \min E > \omega_k(E)\}| \\ &=\; |\{E \subset \{1,2,\ldots,n\} \,:\, E = \emptyset \text{ or } \min E > |E|\}| \\ &=\; F_{n+1}, \end{aligned}$$

where the last equality follows from [5, Theorem 1]. □

Proof of Theorem 1.2. We consider the three cases in the theorem.
When $k = 1$ and $\ell \geqslant 0$,

$$\mathcal{A}_{1,1+\ell} \;=\; \{\emptyset\} \cup \{E \in \mathcal{N} \,:\, E \neq \emptyset,\ \max E \leqslant 1+\ell,\ \text{and } \min E > \omega_1(E)\}.$$

Hence, for a nonempty set $E \in \mathcal{A}_{1,1+\ell}$, $1 \in E$ implies that

$$1 \;=\; \min E \;>\; \omega_1(E) \;=\; |E| - 1,$$

which gives $|E| < 2$. Therefore, $1 \in E$ implies that $E = \{1\}$. We can write $\mathcal{A}_{1,1+\ell}$ as

$$\begin{aligned} &\{\emptyset\} \cup \{\{1\}\} \cup \{E \subset \{1,2,\ldots,1+\ell\} \,:\, E \neq \emptyset \text{ and } \min E > |E|\} \\ =\ &\{\{1\}\} \cup \{E \subset \{1,2,\ldots,1+\ell\} \,:\, E = \emptyset \text{ or } \min E > |E|\}. \end{aligned}$$

We use [5, Theorem 1] to obtain $|\mathcal{A}_{1,1+\ell}| = F_{\ell+2} + 1$.
When $k \geqslant 2$ and $-k < \ell < 0$, that $a_{k,k+\ell} = F_{k+\ell+1}$ follows from Proposition 3.5.
When $k \geqslant 2$ and $\ell \geqslant 0$, we apply Lemma 3.2 with

$$\begin{cases} \mathbf{s}_2 & = \ (a_{\ell+2,\ell+2})_{\ell \geqslant 0}, \\ (t^{\mathbf{s}_2}_{k-1,\ell})_{\ell \geqslant 0} & = \ (a_{k,k+\ell})_{\ell \geqslant 0} \text{ (due to Proposition 3.1)}, \\ (t^{\mathbf{s}_1}_{k-1,\ell})_{\ell \geqslant 0} & = \ (P^{(k-1)}((a_{1,j+1})_{j \geqslant 0})(\ell))_{\ell \geqslant 0}, \end{cases}$$

to obtain

$$a_{k,k+\ell} - P^{(k-1)}((a_{1,j+1})_{j \geqslant 0})(\ell) \;=\; \sum_{i=0}^{k-2} \binom{\ell}{i} a_{k-i,k-i}. \tag{3.7}$$

By Theorem 1.1 and Proposition 3.5, we can write (3.7) as

$$a_{k,k+\ell} - P^{(k-1)}((F_{j+2} + 1)_{j \geqslant 0})(\ell) \;=\; 2\sum_{i=0}^{k-2} \binom{\ell}{i} F_{k-i}.$$

Using Lemma 3.3, we arrive at

$$a_{k,k+\ell} \;=\; 2\sum_{i=0}^{k-2} \binom{\ell}{i} F_{k-i} + \binom{\ell}{k-1} + P^{(k-1)}((F_{j+2})_{j \geqslant 0})(\ell),$$

which, due to Lemma 3.4 and then (3.2), gives

$$\begin{aligned} a_{k,k+\ell} &= 2\sum_{i=0}^{k-2}\binom{\ell}{i}F_{k-i} + \binom{\ell}{k-1} + P^{(k-1)}((F_j)_{j\geqslant 0})(\ell) + P^{(k-1)}((F_j)_{j\geqslant 0})(\ell+1) \\ &= 2\sum_{i=0}^{k-2}\binom{\ell}{i}F_{k-i} + \binom{\ell}{k-1} + \sum_{j=0}^{\ell-1}\binom{j}{\ell-2-j+k} + \sum_{j=0}^{\ell}\binom{j}{\ell-1-j+k} \\ &= 2\sum_{i=0}^{k-2}\binom{\ell}{i}F_{k-i} + 2\binom{\ell}{k-1} + \sum_{j=0}^{\ell-1}\binom{j+1}{\ell-1-j+k} \\ &= 2\sum_{i=0}^{k-2}\binom{\ell}{i}F_{k-i} + 2\binom{\ell}{k-1} + \sum_{j=1}^{\ell}\binom{j}{\ell-j+k}, \end{aligned}$$

as desired. □

Proof of Theorem 1.4. The proof follows immediately from Theorem 1.2 and the identity

$$\sum_{i=0}^{\ell}\binom{\ell}{i}F_{k-i} \;=\; F_{k+\ell} \text{ for all } \ell \geqslant 0 \text{ and } k \geqslant \ell+2, \tag{3.8}$$

which we prove in the appendix. □

4. Appendix

Proof of (1.2). For $k \leqslant 1$, let

$$\begin{aligned} \mathcal{A} &:= \{E \in \mathcal{N} : E = \emptyset \text{ or } \min E > \omega_k(E)\} \text{ and} \\ \mathcal{B} &:= \mathcal{S}^{NMAX} \cup \{E \in \mathcal{S}^{MAX} : k \in E\}. \end{aligned}$$

We shall show that $\mathcal{A} = \mathcal{B}$. Let us rewrite $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\begin{aligned} \mathcal{A}_1 &= \{E \in \mathcal{N} : k \in E \text{ and } \min E > |E|-1\} \text{ and} \\ \mathcal{A}_2 &= \{E \in \mathcal{N} : (k \notin E \text{ and } \min E > |E|) \text{ or } E = \emptyset\}. \end{aligned}$$

First, we show $\mathcal{A} \subset \mathcal{B}$. Pick F in $\mathcal{A}_1$. If $F \in \mathcal{S}^{MAX}$, then $F \in \{E \in \mathcal{S}^{MAX} : k \in E\} \subset \mathcal{B}$. If $F \in S^{NMAX}$, then $F \in \mathcal{B}$ by definition. Hence, $\mathcal{A}_1 \subset \mathcal{B}$. Furthermore, $\mathcal{A}_2$ is a subset of $\mathcal{S}^{NMAX}$. Therefore, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{B}$.

Next, we prove that $\mathcal{B} \subset \mathcal{A}$ by writing

$$\begin{aligned} S^{NMAX} &= \{E \in \mathcal{N} : \min E > |E| \text{ or } E = \emptyset\} \\ &= \{E \in \mathcal{N} : (k \notin E \text{ and } \min E > |E|) \text{ or } E = \emptyset\} \\ &\quad \cup \{E \in \mathcal{N} : k \in E \text{ and } \min E > |E|\}, \end{aligned}$$

which is clearly in $\mathcal{A}_1 \cup \mathcal{A}_2$. Moreover,

$$\{E \in \mathcal{S}^{MAX} : k \in E\} \;=\; \{E \in \mathcal{N} : k \in E \text{ and } \min E = |E|\} \;\subset\; \mathcal{A}_1.$$

Therefore, $\mathcal{B} \subset \mathcal{A}$, which completes our proof. □

Proof of Lemma 3.4. First, we induct on k: for $\ell \geqslant 0$, we have

$$\begin{cases} P^{(0)}((F_{n+2})_{n\geqslant 0})(\ell) & = F_{\ell+2} \\ P^{(0)}((F_n)_{n\geqslant 0})(\ell) + P^{(0)}((F_n)_{n\geqslant 0})(\ell+1) & = F_\ell + F_{\ell+1} \;=\; F_{\ell+2}. \end{cases}$$

Hence, (3.6) holds for $k=0$. Inductive hypothesis: suppose that (3.6) is true for $k\geqslant 0$. We show that it is true for $k+1$, i.e.,

$$P^{(k+1)}((F_{n+2})_{n\geqslant 0})(\ell) \;=\; P^{(k+1)}((F_n)_{n\geqslant 0})(\ell)+P^{(k+1)}((F_n)_{n\geqslant 0})(\ell+1) \text{ for all } \ell\geqslant 0. \quad (4.1)$$

We induct on ℓ. Clearly, (4.1) is true for $\ell=0$, because then both sides are 0. Inductive hypothesis: suppose that (4.1) holds for some $\ell\geqslant 0$. By the inductive hypotheses, we write

$$\begin{aligned}
P^{(k+1)}((F_{n+2})_{n\geqslant 0})(\ell+1) &= P^{(k+1)}((F_{n+2})_{n\geqslant 0})(\ell)+P^{(k)}((F_{n+2})_{n\geqslant 0})(\ell)\\
&= (P^{(k+1)}((F_n)_{n\geqslant 0})(\ell)+P^{(k+1)}((F_n)_{n\geqslant 0})(\ell+1))+\\
&\quad (P^{(k)}((F_n)_{n\geqslant 0})(\ell)+P^{(k)}((F_n)_{n\geqslant 0})(\ell+1))\\
&= (P^{(k+1)}((F_n)_{n\geqslant 0})(\ell)+P^{(k)}((F_n)_{n\geqslant 0})(\ell))+\\
&\quad (P^{(k+1)}((F_n)_{n\geqslant 0})(\ell+1)+P^{(k)}((F_n)_{n\geqslant 0})(\ell+1))\\
&= P^{(k+1)}((F_n)_{n\geqslant 0})(\ell+1)+P^{(k+1)}((F_n)_{n\geqslant 0})(\ell+2),
\end{aligned}$$

which finishes our proof of (4.1). □

Proof of (3.8). We prove by induction on ℓ. Base case: (3.8) clearly holds for $\ell=0$. Inductive hypothesis: suppose that (3.8) holds for some $\ell\geqslant 0$. We show that it holds for $\ell+1$. We have

$$\begin{aligned}
\sum_{i=0}^{\ell+1}\binom{\ell+1}{i}F_{k-i} &= F_k+F_{k-\ell-1}+\sum_{i=1}^{\ell}\binom{\ell+1}{i}F_{k-i}\\
&= F_k+F_{k-\ell-1}+\sum_{i=1}^{\ell}\binom{\ell}{i}F_{k-i}+\sum_{i=1}^{\ell}\binom{\ell}{i-1}F_{k-i}\\
&= \sum_{i=0}^{\ell}\binom{\ell}{i}F_{k-i}+\sum_{i=0}^{\ell}\binom{\ell}{i}F_{k-1-i}\\
&= \left(\sum_{i=0}^{\ell}\binom{\ell}{i}F_{k+1-i}-\sum_{i=0}^{\ell}\binom{\ell}{i}F_{k-1-i}\right)+\sum_{i=0}^{\ell}\binom{\ell}{i}F_{k-1-i}\\
&= \sum_{i=0}^{\ell}\binom{\ell}{i}F_{k+1-i} \;=\; F_{k+1+\ell},
\end{aligned}$$

where the last equality is due to the inductive hypothesis. This completes our proof. □

Acknowledgement

The authors would like to thank the anonymous referee for helpful suggestions that helped improve the paper's exposition.

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MSC2020: 11Y55, 11B37

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