A NEW CLASSIFICATION OF THE KAPREKAR NUMBERS

HARUO IWASAKI

Abstract. Five sets of Kaprekar numbers are exhibited. A nonzero n-digit number is shown to be a Kaprekar number if and only if it is a member of one of the given sets. Our result gives a new classification of the Kaprekar numbers.

1. Introduction

Let D_n be the set of n-digit nonnegative integers and set $D = \bigcup_{n=0}^{\infty} D_n$ ($D_0 = \{0\}$). There is more than one notion of a 'Kaprekar number'. Let us make clear what we mean.

Definition 1.1. *Given a member $\alpha = a_1 \cdots a_n$ of D, reordering $a_1, \ldots, a_n$ in descending order, if necessary, we obtain α_M, and reversing the order of the digits of the latter, we obtain α_L. Let $f(\alpha) = \alpha_M - \alpha_L$. The mapping f from D to itself is called the* Kaprekar transformation. *A nonzero member α of D such that $f(\alpha) = \alpha$ is called a* Kaprekar number.

For example, $f(6174) = 7641 - 1467 = 6174$. D. R. Kaprekar [2] noticed that any member α of D_4 is sent to, by successive applications of the Kaprekar transformation, either 0 or 6174. A similar phenomenon is observed for D_3, where the role played by 6174 in D_4 is replaced by 495.

Definition 1.2. *A Kaprekar number κ in D_n is called a* Kaprekar constant *if any member of D_n is sent to, by successive applications of Kaprekar transformation, either* 0 *or κ.*

Prichett, et al. [3] showed that 6174 and 495 are the only Kaprekar constants. Meanwhile, Hirata [1] found, by computation, 257 Kaprekar numbers less than or equal to 10^{31}.

In this paper, we present five sets of mutually disjoint Kaprekar numbers, T_1, T_2, T_3, T_4, and T_5, and show that $T = \bigcup_{n=1}^{5} T_n$ is the set of all the Kaprekar numbers. Meanwhile, the paper by Prichett, et al. [3] contains a proof showing that certain classes, A, B, C, and D, of sets of numbers give rise to a complete classification of the Kaprekar numbers. (The definitions of the latter classes shall be given later.) Our result, however, is explicit and gives a simple method to obtain all the Kaprekar numbers in a given D_n. Our result also provides a proof for the nonexistence of Kaprekar constants except 6174 and 495. We shall show that our classification of the Kaprekar numbers and the classification due to Prichett, et al. [3] are equivalent.

2. Five Sets of Kaprekar Numbers

Every member of D_1 is sent to 0 by the Kaprekar transformation. A member α of D_2 is equal to α_M or α_L, hence $f(\alpha) = \alpha$ implies that $\alpha = 0$. Therefore, the set D_2 contains no Kaprekar number. Therefore, for our purpose of finding Kaprekar numbers in D_n, we may assume that $n \geq 3$.

Definition 2.1.

(1) $T_1 = \{f(\underbrace{\alpha_1 \cdots \alpha_1}_{x_1});\ x_1 \in \mathbb{N}\}$ $(\alpha_1 = 495)$

(2) $T_2 = \{f(6174\underbrace{\alpha_2 \cdots \alpha_2}_{y_2});\ y_2 \in \mathbb{N} \cup \{0\}\}$ $(\alpha_2 = 36)$

(3) $T_3 = \{f(\underbrace{\alpha_3 \cdots \alpha_3}_{x_3}\underbrace{\alpha_2 \cdots \alpha_2}_{y_3});\ x_3 \in \mathbb{N},\ y_3 \in \mathbb{N} \cup \{0\}\}$ $(\alpha_3 = 123456789)$

(4) $T_4 = \{f(\underbrace{\alpha_3 \cdots \alpha_3}_{x_{4,1}}\underbrace{\alpha_2 \cdots \alpha_2}_{x_{4,2}}\underbrace{\alpha_1 \cdots \alpha_1}_{2x_{4,2}}\underbrace{\alpha_4 \cdots \alpha_4}_{3x_{4,2}});\ x_{4,1}, x_{4,2} \in \mathbb{N}\}$ $(\alpha_4 = 27)$

(5) $T_5 = \{f(\underbrace{\alpha_5 \cdots \alpha_5}_{x_{5,1}}\underbrace{\alpha_6 \cdots \alpha_6}_{x_{5,2}}\underbrace{\alpha_3 \cdots \alpha_3}_{y_{5,1}}\underbrace{\alpha_2 \cdots \alpha_2}_{y_{5,2}});\ x_{5,1}, x_{5,2} \in \mathbb{N},\ y_{5,1}, y_{5,2} \in \mathbb{N}\cup\{0\}\}$ $(\alpha_5 = 124578,\ \alpha_6 = 09)$

To see that the elements of T_1 are Kaprekar numbers, we set $\alpha = \underbrace{\alpha_1 \cdots \alpha_1}_{x_1}$. We have $\alpha_M = \underbrace{9 \cdots 9}_{x_1}\underbrace{5 \cdots 5}_{x_1}\underbrace{4 \cdots 4}_{x_1}$ and $f(\alpha) = \underbrace{5 \cdots 5}_{x_1-1}4\underbrace{9 \cdots 9}_{x_1}\underbrace{4 \cdots 4}_{x_1-1}5$. We see that the numbers of the digits appearing in $f(\alpha)$ and α_M are the same. This implies that $f(\alpha) = \alpha$. The same argument applies to show that the elements of T_2, T_3, T_4 and T_5 are Kaprekar numbers.

The least digit appearing in $\alpha \in T_1$ is 4. For members of T_2, T_3, and T_4, the least digit is 1. The members of T_5 have least digit 0. The maximum digit in a member of T_2 is 7, whereas T_1, T_3, T_4, and T_5 all have a maximum digit of 9. For the members of T_3, the number of 9 digits does not exceed that of 6, whereas the number of 9 digits in T_4 exceeds that of the digit 6. Hence, the sets T_1, T_2, T_3, T_4, and T_5 are mutually disjoint.

3. The Classification by Prichett, et al.

Suppose we have $\alpha \in D_n$ $(n \geq 3)$. Let us write $\alpha_M = b_n \cdots b_1$ and $\alpha_L = c_n \cdots c_1$ and suppose that $\alpha_M > \alpha_L$. We then have $f(\alpha) = \alpha_M - \alpha_L = a_1 \cdots a_k b \underbrace{9 \cdots 9}_{l} c a'_k \cdots a'_1$ $(k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\})$; $a_1 + a'_1 = 10$, $a_i + a'_i = 9$ $(2 \leq i \leq k)$, $b + c = 8$; $a_1 \geq \cdots \geq a_k \geq b$, and if $k \geq 2$, $c \geq a'_k \geq \cdots \geq a'_2$.

Following Prichett, et al. [3], we set $n = 2r + \delta$ $(\delta = 0, 1)$. Let us write $d_n = b_n - c_n$, $\ldots$, $d_r = b_r - c_r$ and $d(\alpha) = d_n \cdots d_r$. For instance, if $\alpha = 6174$, we have $n = 4$, $r = 2$, and $d(\alpha) = 62$; whereas, for $\beta = 495$ we have $n = 3$, $r = 1$, and $d(\beta) = 5$. As Prichett, et al. point out, $f(\alpha)$ is readily obtained from $d(\alpha)$. They also proved that an element $\alpha \in D$ is a Kaprekar number if and only if the corresponding $d(\alpha)$ belongs to one of four classes described below.

For a given member of D_n $(n = 2r + \delta)$ and a digit a $(a = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, we denote by l_a the number of a's appearing in the given member of D_n. We consider the following classes, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$.

$\mathcal{A}$: $l_8 = l_6 = l_4 = l_2 = 2l_0 + \delta$, $l_7 = l_5 = l_1$, $l_9 = 0$ if and only if $l_1 = 0$, l_0, l_1, or δ is nonzero.

$\mathcal{B}$: $l_9 = l_1 = 0$, $l_6 = l_8$, $l_7 = 2l_3$, $l_4 = l_2 = l_3 + l_8$, $l_5 = l_3 \neq 0$, and $l_0 = l_3 + (l_8 - \delta)/2$.

$\mathcal{C}$: $l_6 = l_2 = 1$, $l_i = 0 \ \ (i \neq 2, 3, 6)$, and $\delta = 0$.

$\mathcal{D}$: $l_5 = 2l_0 + \delta$, $l_i = 0 \ \ (i \neq 0, 5)$, l_0, or δ is nonzero.

The original paper by Prichett, et al. [3] contained errors with regard to $\mathcal{B}$; the above is the version corrected by the referee. We added '$\neq$' following $l_5 = l_3$ in the definition for $\mathcal{B}$ to avoid $\mathcal{A}$ and $\mathcal{B}$ sharing elements. If an element α in $\mathcal{B}$ satisfies, instead of $l_5 = l_3 \neq 0$, the requirement $l_5 = l_3 = 0$, we must have $l_7 = 0$, $l_9 = l_1 = 0$, $l_4 = l_2 = l_8 = l_6 = 2l_0 + \delta$, and furthermore, $l_7 = l_5 = l_1 = 0$. In addition, if $l_0 = \delta = 0$, we end up having $l_i = 0$ for all $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, which is impossible. Thus, if the condition $l_5 = l_3 \geq 0$ is left as in the original, the specific case of $l_5 = l_3 = 0$ for α in $\mathcal{B}$ implies that at least one of l_0 and δ must be nonzero, thus, $\alpha \in \mathcal{A}$.

Theorem 3.1. *Let α be a member of D_n $(n \geq 3)$. The digit α is a Kaprekar number if and only if it belongs to one of T_i $(1 \leq i \leq 5)$.*

Proof. Because we have already shown that every member of T_i $(1 \leq i \leq 5)$ is a Kaprekar number, it is enough to show that every member of the $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ belongs to one of the latter T_i $(1 \leq i \leq 5)$.

Let us denote $\alpha = a_1 \cdots a_n$ of D_n with $n = 2r + \delta$, $(\delta = 0, 1)$, and $\tilde{\alpha} = a_1 \cdots a_r$.

We begin by computing, for each member α of the above T_i, its $d(\alpha)$.

(1) $\alpha = f(\underbrace{\alpha_1 \cdots \alpha_1}_{x}) \in T_1$, $\alpha_1 = 495$, and $x \in \mathbb{N}$. We have $n = 3x$.

(1-1) $x = 2x'$. We then have $r = 3x'$ and $\delta = 0$.

$\widetilde{\alpha_M} = \underbrace{9 \cdots 9}_{x}\underbrace{5 \cdots 5}_{x'}$, $\widetilde{\alpha_L} = \underbrace{4 \cdots 4}_{x}\underbrace{5 \cdots 5}_{x'}$. Hence, for $\alpha \in T_1$ in the present case, we have $d(\alpha) = \underbrace{5 \cdots 5}_{x}\underbrace{0 \cdots 0}_{x'}$.

(1-2) $x = 2x' + 1$, $r = 3x'$, $\delta = 1$, and $x' \geq 0$.

We have, for $\alpha \in T_1$ in the latter case, $d(\alpha) = \underbrace{5 \cdots 5}_{x}\underbrace{0 \cdots 0}_{x'}$.

(2) $\alpha = f(6174\underbrace{\alpha_2 \cdots \alpha_2}_{y}) \in T_2$, $\alpha_2 = 36$, and $y \in \mathbb{N} \cup \{0\}$. We have $n = 4 + 2y$, $r = 2 + y$, and $\delta = 0$.

$\widetilde{\alpha_M} = 7\underbrace{6 \cdots 6}_{y+1}$, $\widetilde{\alpha_L} = 1\underbrace{3 \cdots 3}_{y}4$, therefore, for $\alpha \in T_2$, we have $d(\alpha) = 6\underbrace{3 \cdots 3}_{y}2$.

(3) $\alpha = f(\underbrace{\alpha_3\cdots\alpha_3}_{x}\underbrace{\alpha_2\cdots\alpha_2}_{y}) \in T_3$, $\alpha_3 = 123456789$, $x \in \mathbb{N}$, and $y \in \mathbb{N}\cup\{0\}$. We have $n = 9x + 2y$.

(3-1) $x = 2x'$ and $r = 9x' + y$.

$\widetilde{\alpha_M} = \underbrace{9\cdots9}_{x}\underbrace{8\cdots8}_{x}\underbrace{7\cdots7}_{x}\underbrace{6\cdots6}_{x+y}\underbrace{5\cdots5}_{x'}$ and $\widetilde{\alpha_L} = \underbrace{1\cdots1}_{x}\underbrace{2\cdots2}_{x}\underbrace{3\cdots3}_{x+y}\underbrace{4\cdots4}_{x}\underbrace{5\cdots5}_{x'}$.

Therefore, for $\alpha \in T_3$ in the present case, we have

$d(\alpha) = \underbrace{8\cdots8}_{x}\underbrace{6\cdots6}_{x}\underbrace{4\cdots4}_{x}\underbrace{3\cdots3}_{y}\underbrace{2\cdots2}_{x}\underbrace{0\cdots0}_{x'}$.

(3-2) $x = 2x' + 1$, $r = 9x' + 4 + y$, $n = 2r + \delta$, and $\delta = 1$.

$\widetilde{\alpha_M} = \underbrace{9\cdots9}_{x}\underbrace{8\cdots8}_{x}\underbrace{7\cdots7}_{x}\underbrace{6\cdots6}_{x+y}\underbrace{5\cdots5}_{x'}$ and $\widetilde{\alpha_L} = \underbrace{1\cdots1}_{x}\underbrace{2\cdots2}_{x}\underbrace{3\cdots3}_{x+y}\underbrace{4\cdots4}_{x}\underbrace{5\cdots5}_{x'}$.

Therefore, for $\alpha \in T_3$ in the present case, we have

$d(\alpha) = \underbrace{8\cdots8}_{x}\underbrace{6\cdots6}_{x}\underbrace{4\cdots4}_{x}\underbrace{3\cdots3}_{y}\underbrace{2\cdots2}_{x}\underbrace{0\cdots0}_{x'}$.

(4) $\alpha = f(\underbrace{\alpha_3\cdots\alpha_3}_{x_1}\underbrace{\alpha_2\cdots\alpha_2}_{x_2}\underbrace{\alpha_1\cdots\alpha_1}_{2x_2}\underbrace{\alpha_4\cdots\alpha_4}_{3x_2}) \in T_4$ and $\alpha_4 = 27, x_1, x_2 \in \mathbb{N}$. We have $n = 9x_1 + 14x_2 = 2r + \delta$.

(4-1) $x_1 = 2x_1'$ and $r = 9x_1' + 7x_2$.

$\widetilde{\alpha_M} = \underbrace{9\cdots9}_{x_1+2x_2}\underbrace{8\cdots8}_{x_1}\underbrace{7\cdots7}_{x_1+3x_2}\underbrace{6\cdots6}_{x_1+x_2}\underbrace{5\cdots5}_{x_1'+x_2}$ and $\widetilde{\alpha_L} = \underbrace{1\cdots1}_{x_1}\underbrace{2\cdots2}_{x_1+3x_2}\underbrace{3\cdots3}_{x_1+x_2}\underbrace{4\cdots4}_{x_1+2x_2}\underbrace{5\cdots5}_{x_1'+x_2}$.

Therefore, for $\alpha \in T_4$ in the present case, we have

$d(\alpha) = \underbrace{8\cdots8}_{x_1}\underbrace{7\cdots7}_{2x_2}\underbrace{6\cdots6}_{x_1}\underbrace{5\cdots5}_{x_2}\underbrace{4\cdots4}_{x_1+x_2}\underbrace{3\cdots3}_{x_2}\underbrace{2\cdots2}_{x_1+x_2}\underbrace{0\cdots0}_{x_1'+x_2}$.

(4-2) $x_1 = 2x_1' + 1$, $n = 2r + 1$, and $r = 9x_1' + 7x_2 + 4$.

$\widetilde{\alpha_M} = \underbrace{9\cdots9}_{x_1+2x_2}\underbrace{8\cdots8}_{x_1}\underbrace{7\cdots7}_{x_1+3x_2}\underbrace{6\cdots6}_{x_1+x_2}\underbrace{5\cdots5}_{x_1'+x_2}$.

$\widetilde{\alpha_L} = \underbrace{1\cdots1}_{x_1}\underbrace{2\cdots2}_{x_1+3x_2}\underbrace{3\cdots3}_{x_1+x_2}\underbrace{4\cdots4}_{x_1+2x_2}\underbrace{5\cdots5}_{x_1'+x_2}$.

Therefore, for $\alpha \in T_4$ in the present case, we have

$d(\alpha) = \underbrace{8\cdots8}_{x_1}\underbrace{7\cdots7}_{2x_2}\underbrace{6\cdots6}_{x_1}\underbrace{5\cdots5}_{x_2}\underbrace{4\cdots4}_{x_1+x_2}\underbrace{3\cdots3}_{x_2}\underbrace{2\cdots2}_{x_1+x_2}\underbrace{0\cdots0}_{x_1'+x_2}$.

(5) $\alpha = f(\underbrace{\alpha_5 \cdots \alpha_5}_{x_1}\underbrace{\alpha_6 \cdots \alpha_6}_{x_2}\underbrace{\alpha_3 \cdots \alpha_3}_{y_1}\underbrace{\alpha_2 \cdots \alpha_2}_{y_2}) \in T_5$, $\alpha_5 = 124578$, $\alpha_6 = 09$, $x_1, x_2 \in \mathbb{N}$, and $y_1, y_2 \in \mathbb{N} \cup \{0\}$.

We have $n = 6x_1 + 2x_2 + 9y_1 + 2y_2 = 2(3x_1 + x_2 + 4y_1 + y_2) + y_1 = 2r + \delta$.

(5-1) $y_1 = 2y_1'$, $r = 3x_1 + x_2 + 4y_1 + y_2 + y_1'$, and $\delta = 0$.

$\widetilde{\alpha_M} = \underbrace{9 \cdots 9}_{x_2+y_1}\underbrace{8 \cdots 8}_{x_1+y_1}\underbrace{7 \cdots 7}_{x_1+y_1}\underbrace{6 \cdots 6}_{y_1+y_2}\underbrace{5 \cdots 5}_{x_1+y_1'}$.

$\widetilde{\alpha_L} = \underbrace{0 \cdots 0}_{x_2}\underbrace{1 \cdots 1}_{x_1+y_1}\underbrace{2 \cdots 2}_{x_1+y_1}\underbrace{3 \cdots 3}_{y_1+y_2}\underbrace{4 \cdots 4}_{x_1+y_1}\underbrace{5 \cdots 5}_{y_1'}$.

Therefore, for $\alpha \in T_5$ in the present case, we have

$d(\alpha) = \underbrace{9 \cdots 9}_{x_2}\underbrace{8 \cdots 8}_{y_1}\underbrace{7 \cdots 7}_{x_1}\underbrace{6 \cdots 6}_{y_1}\underbrace{5 \cdots 5}_{x_1}\underbrace{4 \cdots 4}_{y_1}\underbrace{3 \cdots 3}_{y_2}\underbrace{2 \cdots 2}_{y_1}\underbrace{1 \cdots 1}_{x_1}\underbrace{0 \cdots 0}_{y_1'}$.

(5-2) $y_1 = 2y_1' + 1$ and $r = 3x_1 + x_2 * 4y_1 + y_2 + y_1' + 4$.

$\widetilde{\alpha_M} = \underbrace{9 \cdots 9}_{x_2+y_1}\underbrace{8 \cdots 8}_{x_1+y_1}\underbrace{7 \cdots 7}_{x_1+y_1}\underbrace{6 \cdots 6}_{y_1+y_2}\underbrace{5 \cdots 5}_{x_1+y_1'}$.

$\widetilde{\alpha_L} = \underbrace{0 \cdots 0}_{x_2}\underbrace{1 \cdots 1}_{x_1+y_1}\underbrace{2 \cdots 2}_{x_1+y_1}\underbrace{3 \cdots 3}_{y_1+y_2}\underbrace{4 \cdots 4}_{x_1+y_1}\underbrace{5 \cdots 5}_{y_1'}$.

Therefore, for $\alpha \in T_5$ in the present case, we have

$d(\alpha) = \underbrace{9 \cdots 9}_{x_2}\underbrace{8 \cdots 8}_{y_1}\underbrace{7 \cdots 7}_{x_1}\underbrace{6 \cdots 6}_{y_1}\underbrace{5 \cdots 5}_{x_1}\underbrace{4 \cdots 4}_{y_1}\underbrace{3 \cdots 3}_{y_2}\underbrace{2 \cdots 2}_{y_1}\underbrace{1 \cdots 1}_{x_1}\underbrace{0 \cdots 0}_{y_1'}$.

Now, we show that every member belonging to one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$ is contained in one of T_1, T_2, T_3, T_4, or T_5.

(a) $\alpha \in \mathcal{A}$. We have three cases: (a-1), (a-2), and (a-3).

(a-1) $l_0 \neq 0$. In this case, we have $l_8 = l_6 = l_4 = l_2 = 2l_0 + \delta = x \neq 0$. Let $l_0 = x'$.

Then an element α of $\mathcal{A}$ of the latter case (a-1), with

$d(\alpha) = \underbrace{8 \cdots 8}_{x}\underbrace{6 \cdots 6}_{x}\underbrace{4 \cdots 4}_{x}\underbrace{3 \cdots 3}_{y}\underbrace{2 \cdots 2}_{x}\underbrace{0 \cdots 0}_{x'}$ $(y \in \mathbb{N} \cup \{0\})$ is an element of T_3.

(a-2) $l_1 \neq 0$. In this case, we have $l_9 \neq 0$, $l_7 = l_5 = l_1$, and $l_8 = l_6 = l_4 = l_2 = 2l_0 + \delta$.

Let $l_1 = x_1$, $l_9 = x_2$, and $l_8 = l_6 = l_4 = l_2 = 2l_0 + \delta = y_1$, $l_0 = y_1'$, and $l_3 = y_2$. We then have $y_1 = 2y_1' + \delta$ and $x_1, x_2 \in \mathbb{N}$, and $y_1, y_1', y_2 \in \mathbb{N} \cup \{0\}$. The element $\alpha \in \mathcal{A}$ with

$$d(\alpha) = \underbrace{9 \cdots 9}_{x_2}\underbrace{8 \cdots 8}_{y_1}\underbrace{7 \cdots 7}_{x_1}\underbrace{6 \cdots 6}_{y_1}\underbrace{5 \cdots 5}_{x_1}\underbrace{4 \cdots 4}_{y_1}\underbrace{3 \cdots 3}_{y_2}\underbrace{2 \cdots 2}_{y_1}\underbrace{1 \cdots 1}_{x_1}\underbrace{0 \cdots 0}_{y_1'}$$

belongs to T_5.

(a-3) $\delta \neq 0$. Because we have already addressed the cases $l_0 \neq 0$ and $l_1 \neq 0$, we may assume that $l_0 = l_1 = 0$. We now have $l_8 = l_6 = l_4 = l_2 = \delta = 1$, $l_7 = l_5 = l_1 = 0$, and $l_0 = L_9 = 0$. Hence, the element $\alpha = f(\alpha_3) \in T_3$.

Therefore, we have $\mathcal{A} \subset T_3 \cup T_5$.

(b) $\alpha \in \mathcal{B}$. From the definition of $\mathcal{B}$ and the computations in (4) above, we have $\alpha \in T_4$.

(c) $\alpha \in \mathcal{C}$. We have $\alpha \in T_2$.

(d) $\alpha \in \mathcal{D}$. We have $\alpha \in T_1$.

Hence, every member of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ is contained in one of T_1, T_2, T_3, T_4, and T_5. Therefore, a member α of D_n $(n \geq 3)$ is a Kaprekar number if and only if it belongs to one of T_i $(1 \leq i \leq 5)$. $\square$

4. Kaprekar Numbers in D_n

Let the set of Kaprekar numbers of degree n be denoted by K_n. We have shown that $K_1 = K_2 = \emptyset$ and $K_3 = \{495\}$. For $n \geq 4$, $K_n = D_n \cap T$ $(T = \bigcup_{1\leq i\leq 5} T_i)$.

For the sake of simplicity, let

$\kappa_1(x_1) = f(\underbrace{\alpha_1 \cdots \alpha_1}_{x_1})$ $(\alpha_1 = 495)$.

$\kappa_2(y_2) = f(6174\,\underbrace{\alpha_2 \cdots \alpha_2}_{y_2})$ $(\alpha_2 = 36)$.

$\kappa_3(x_3, y_3) = f(\underbrace{\alpha_3 \cdots \alpha_3}_{x_3}\underbrace{\alpha_2 \cdots \alpha_2}_{y_3})$ $(\alpha_3 = 123456789)$.

$\kappa_4(x_{4,1}, x_{4,2}) = f(\underbrace{\alpha_3 \cdots \alpha_3}_{x_{4,1}}\underbrace{\alpha_2 \cdots \alpha_2}_{x_{4,2}}\underbrace{\alpha_1 \cdots \alpha_1}_{2x_{4,2}}\underbrace{\alpha_4 \cdots \alpha_4}_{3x_{4,2}})$ $(\alpha_4 = 27)$.

$\kappa_5(x_{5,1}, x_{5,2}, y_{5,1}, y_{5,2}) = f(\underbrace{\alpha_5 \cdots \alpha_5}_{x_{5,1}}\underbrace{\alpha_6 \cdots \alpha_6}_{x_{5,2}}\underbrace{\alpha_3 \cdots \alpha_3}_{y_{5,1}}\underbrace{\alpha_2 \cdots \alpha_2}_{y_{5,2}})$ $(\alpha_5 = 124578$ and $\alpha_6 = 09)$.

We have

$T_1 = \{\kappa_1(x_1);\ x_1 \in \mathbb{N}\}$.

$T_2 = \{\kappa_2(y_2);\ y_2 \in \mathbb{N} \cup \{0\}\}$.

$T_3 = \{\kappa_3(x_3, y_3);\ x_3 \in \mathbb{N},\ y_3 \in \mathbb{N} \cup \{0\}\}$.

$T_4 = \{\kappa_4(x_{4,1}, x_{4,2});\ x_{4,1}, x_{4,2} \in \mathbb{N}\}$.

$T_5 = \{\kappa_5(x_{5,1}, x_{5,2}, y_{5,1}, y_{5,2});\ x_{5,1}, x_{5,2} \in \mathbb{N},\ y_{5,1}, y_{5,2} \in \mathbb{N} \cup \{0\}\}$.

To obtain the members of K_n $(n \geq 4)$, we look at the degrees of $\kappa_1(x_1)$, ..., $\kappa_5(x_{5,1}, x_{5,2}, y_{5,1}, y_{5,2})$ and solve the following set of Diophantine equations.

$$n = \begin{cases} 3x_1 & (x_1 \in \mathbb{N}) \quad or \\ 4 + 2y_2 & (y_2 \in \mathbb{N} \cup \{0\}) \quad or \\ 9x_3 + 2y_3 & (x_3 \in \mathbb{N}, y_3 \in \mathbb{N} \cup \{0\}) \quad or \\ 9x_{4,1} + 14x_{4,2} & (x_{4,1}, x_{4,2} \in \mathbb{N}) \quad or \\ 6x_{5,1} + 2x_{5,2} + 9y_{5,1} + 2y_{5,2} & (x_{5,1}, x_{5,2} \in \mathbb{N}, y_{5,1}, y_{5,2} \in \mathbb{N} \cup \{0\}). \end{cases} \tag{4.1}$$

For instance, the only solution of the above set of equations for $n = 4$ is $n = 4 + 2 \times 0$. Therefore, $K_4 = \{6174\}$. Similarly, we have
$K_5 = \emptyset$, $K_6 = \{\kappa_1(2), \kappa_2(1)\}$, $K_7 = \emptyset$, and $K_8 = \{\kappa_2(2),\ \kappa_5(1, 1, 0, 0)\}$.

For an even $8 + 2k$ $(k \geq 0)$, we have $K_{8+2k} \ni \kappa_2(2 + k)$, $\kappa_5(1, 1, 0, k)$, and therefore, D_{2n} $(n > 2)$ may not have a Kaprekar constant. Consider K_{2n+1} $(n \geq 4)$. Again, by examining equation (4.1), we have, $K_9 = \{\kappa_1(3), \kappa_3(1, 0)\}$, $K_{11} = \{\kappa_3(1, 1)\}$, $K_{13} = \{\kappa_3(1, 2)\}$. Furthermore, we have $K_{15} = \{\kappa_1(5), \kappa_3(1, 3)\}$ and for $2n + 1$ with $n = 8 + k$ and $k \geq 0$, we have $K_{2n+1} \ni \kappa_3(1, 4 + k)$, $\kappa_5(1, 1 + k, 1, 0)$. Hence, for all D_{2n+1} $(n \geq 2)$ except D_{11} and D_{13}, we note that D_{2n+1} contains no Kaprekar constant.

Consider the above exceptional cases. We note that an element of D_{11}, $\alpha = 86420987532$, and an element $\beta = 876532664322$ of D_{13} satisfy $f^5(\alpha) = \alpha$ with $f^k(\alpha) \neq \alpha$ for $k < 5$; whereas $f^2(\beta) = \beta$ with $f(\beta) \neq \beta$. Thus, α never reaches, by successive applications of the Kaprekar transformation f, to $\kappa_3(1, 1)$, the only Kaprekar number in D_{11}, which means that $\kappa_3(1, 1)$ is not a Kaprekar constant. A similar proposition holds for $\kappa_3(1, 2)$ in D_{13}. Therefore, our result contains a verification of the result shown by Prichett, et al., namely, no Kaprekar constants exist except for 495 and 6174.

Acknowledgments

We are grateful to Kenichi Iyanaga, Ph.D., Professor Emeritus at Tokyo University of Marine Sciences, for valuable advice. We are also grateful to the referee of our manuscript.

References

[1] Y. Hirata, *The Kaprekar transformation for higher-digit numbers*, Maebashi Kyoai Gakuen Ronshu, **5** (2005), 21–50.
[2] D. R. Kaprekar, *An interesting property of the number 6174*, Scripta Mathematica **21** (1955), 244–245.
[3] G. D. Prichett, A. L. Ludington, and J. F. Lapenta, *The determination of all decadic Kaprekar constants*, The Fibonacci Quaterly, **19.1** (1981), 45–52.

MSC2020: 11A99, 11D02,11D72

Ranzan Mathematics Study Group, Suwagun Haramura 17217-1472 Nagano-ken Japan 391-0100
Email address: halhal9219@ybb.ne.jp