ON THE EULER FUNCTION OF LINEARLY RECURRENT SEQUENCES

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Abstract. In this paper, we show that if $(U_n)_{n\geq 1}$ is any nondegenerate linearly recurrent sequence of integers whose general term is up to sign not a polynomial in n, then the inequality $\phi(|U_n|) \geq |U_{\phi(n)}|$ holds on a set of positive integers n of density 1, where ϕ is the Euler function. We show that the set of $n \leq x$ for which the above inequality fails has counting function $O_U(x/\log x)$.

1. Introduction

Let $(U_n)_{n\geq 1}$ be a linearly recurrent sequence of integers. Such a sequence satisfies a recurrence of the form

$$U_{n+k} = a_1 U_{n+k-1} + \cdots + a_k U_n \qquad \text{for all } n \geq 1, \tag{1}$$

with integers $a_1, \ldots, a_k$, where $U_1, \ldots, U_k$ are integers. Assuming k is minimal, U_n can be represented as

$$U_n = \sum_{i=1}^{s} P_i(n)\alpha_i^n, \tag{2}$$

where

$$\Psi(X) := X^k - a_1 X^{k-1} - \cdots - a_k = \prod_{i=1}^{s}(X - \alpha_i)^{\sigma_i} \tag{3}$$

is the characteristic polynomial of $(U_n)_{n\geq 1}$, $\alpha_1, \ldots, \alpha_s$ are the distinct roots of $\Psi(X)$ with multiplicities $\sigma_1, \ldots, \sigma_s$, respectively, and $P_i(X)$ is a polynomial of degree $\sigma_i - 1$ with coefficients in $\mathbb{Q}(\alpha_i)$. The sequence is nondegenerate if α_i/α_j is not a root of 1 for any $i \neq j$ in $\{1, \ldots, s\}$. A classic example is the Fibonacci sequence $(F_n)_{n\geq 1}$ that has $k = 2$, $\Psi(X) = X^2 - X - 1$, and initial terms $F_1 = F_2 = 1$. Let $\phi(m)$ and $\sigma(m)$ be the Euler function and sum of divisors function of the positive integer m. In [5], the first author proved that the inequalities

$$\phi(F_n) \geq F_{\phi(n)} \qquad \text{and} \qquad \sigma(F_n) \leq F_{\sigma(n)}$$

hold for all positive integers n. It was also remarked that if instead of considering $(F_n)_{n\geq 1}$, one considers a Lucas sequence with complex conjugated roots, i.e., a nondegenerate binary recurrent sequence $(U_n)_{n\geq 0}$ with $U_0 = 0$, $U_1 = 1$, and $\Psi(X)$ a quadratic polynomial with complex conjugated roots, then the inequality

$$\phi(|U_n|) \geq |U_{\phi(n)}|$$

fails infinitely often. It fails for a positive proportion of prime numbers n. Such questions were recently revisited by other authors (see [4] and [7], for example).

In this paper, we prove the following theorem. Recall that if $f(x)$ and $g(x)$ are functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}_+$, we write $f(x) = O(g(x))$ and $f(x) = o(g(x))$ if the inequality $f(x) < Kg(x)$ holds with some constant $K > 0$ and all $x > x_0$, and $\lim_{x\to\infty} f(x)/g(x) = 0$, respectively. Further, the notations $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are equivalent to

$f(x) = O(g(x))$. When the implied constant K depends on some other parameters like U, ε, we indicate this by writing $f(x) = O_{U,\varepsilon}(g(x))$ or $f(x) \ll_{U,\varepsilon} g(x)$.

Theorem 1. *Let $(U_n)_{n\geq 1}$ be a nondegenerate linearly recurrent sequence of integers such that $|U_n|$ is not a polynomial in n for all large n and let x be a large real number. Then, the inequality*

$$\phi(|U_n|) \geq |U_{\phi(n)}| \tag{4}$$

fails on a set of positive integers $n \leq x$ of cardinality $O_U(x/\log x)$. A similar statement holds for the positive integers $n \leq x$ for which the inequality

$$\sigma(|U_n|) \leq |U_{\sigma(n)}|$$

fails.

The theorem does not hold for sequences for which U_n is either $P(n)$ or $(-1)^n P(n)$, with some polynomial $P(X) \in \mathbb{Z}[X]$, whose characteristic polynomial $\Psi(X)$ is one of $(X-1)^k$ or $(X+1)^k$, where $k-1$ is the degree of $P(X)$. For example, with $k=3$ and $P(X) = X^2+1$, we have that if n is odd, then $U_n = n^2+1$ is even; therefore,

$$\phi(U_n) \leq \frac{n^2+1}{2}.$$

On the other hand, for a positive proportion of n, we have $\phi(n) > n/\sqrt{2}$ and all such n are odd. Indeed, if n is even, then $\phi(n)/n \leq 1/2$, so we cannot have $\phi(n)/n > 1/\sqrt{2}$ for such n. To justify why there are a positive proportion of such n, recall that Schoenberg [9] proved the existence of a continuous monotone function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ such that

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\phi(n)}{n} \leq \alpha \right\} = f(\alpha) \qquad \text{for } \alpha \in [0,1].$$

In particular, the density of the set of n such that $\phi(n)/n > 1/\sqrt{2}$ equals $1 - f(1/\sqrt{2}) > 0$. For such n,

$$U_{\phi(n)} = \phi(n)^2 + 1 > \frac{n^2}{2} + 1 > \phi(U_n).$$

As we said above, the bound $O_U(x/\log x)$ from the statement of Theorem 1 is close to the truth in some cases like when $(U_n)_{n\geq 0}$ is a Lucas sequence with complex conjugated roots. Even more, it is easy to construct binary recurrent sequences $(U_n)_{n\geq 1}$ with real roots for which inequality (4) fails for a number of positive integers $n \leq x$, which is $\gg_U x/(\log x)$. For example, let $q_1 < \cdots < q_k$ be odd primes such that

$$\sum_{i=1}^{k} \frac{1}{q_i} > 1.$$

Let $a > 2$ be a positive integer such that $a \equiv 2 \pmod{q_i}$ for $i = 1, \ldots, k$. Then $2^p - a$ is a multiple of q_i for all $i = 1, \ldots, k$, whenever p is a prime such that $p \equiv 1 \pmod{q_i - 1}$ for all

$i = 1, \dots, k$. For such primes p that are sufficiently large, we have

$$\begin{aligned}\phi(2^p - a) &= (2^p - a) \prod_{q \mid 2^p - a} \left(1 - \frac{1}{q}\right) \le (2^p - a) \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right)\\ &< (2^p - a) \exp\left(-\sum_{i=1}^{k} \frac{1}{q_i}\right) < \frac{2^p - a}{e}\\ &< 2^{p-1} - a = 2^{\phi(p)} - a.\end{aligned}$$

Thus, $\phi(U_n) < U_{\phi(n)}$ for $n = p$ a large prime in the progression

$$p \equiv 1 \pmod{\mathrm{lcm}[q_1 - 1, \dots, q_k - 1]}$$

and $U_n := 2^n - a$, which is the nth term of a binary recurrent sequence of characteristic polynomial $\Psi(X) = X^2 - 3X + 2$. In the above, the notation $\mathrm{lcm}[q_1 - 1, \dots, q_k - 1]$ stands for the least common multiple of $q_1 - 1, \dots, q_k - 1$.

2. Preliminary Results

2.1. **Arithmetic Functions.** Here, we collect a few facts from the anatomy of integers that are useful for our proof of Theorem 1. The first result addresses the minimal order of $\phi(n)$ and the maximal order of $\sigma(n)$. It follows from Theorems 323, 328, and 329 in [3].

Lemma 1. *Let $n \ge 3$. We then have*

$$\frac{\phi(n)}{n} \gg \frac{1}{\log\log n} \qquad \text{and} \qquad \frac{\sigma(n)}{n} \ll \log\log n.$$

For a positive integer n, put $p(n)$ for the smallest prime factor of n with the convention that $p(1) = 1$. For $x \ge y \ge 2$, put

$$\Phi(x, y) := \#\{n \le x : p(n) > y\}.$$

The following inequality is a consequence of the Brun sieve and appears, for example, on page 397 in [10] (see also Exercise on page 11 in [2]).

Lemma 2. *We have, uniformly for $x \ge y \ge 2$,*

$$\Phi(x, y) \ll \frac{x}{\log y}.$$

Let $\Omega(n)$ be the total number of prime factors of n counting multiplicities.

Lemma 3. *Let $x \ge 10$. The number of positive integers $n \le x$ such that $\Omega(n) \ge 10 \log\log x$ is $O(x/(\log x)^2)$.*

Proof. Exercise 05 on page 12 in [2] shows that

$$\#\{n \le x : \Omega(n) = k\} \ll \frac{xk \log x}{2^k}$$

uniformly in $k \geq 1$ and $x \geq 2$. Taking $K := \lfloor 10 \log\log x \rfloor$ and applying the above estimate with $k \geq K$, we get that

$$\begin{aligned} \#\{n \leq x : \Omega(n) \geq 10 \log\log x\} &\ll x \quad \log x \sum_{k \geq K} \frac{k}{2^k} \ll \frac{xK \log x}{2^K} \\ &\ll \quad \frac{x \log x \log\log x}{2^{10 \log\log x}} = \frac{x \log x \log\log x}{(\log x)^{10 \log 2}} \\ &= \quad O\left(\frac{x}{(\log x)^2}\right). \end{aligned}$$

□

Let $\tau(n)$ be the number of divisors of n.

Lemma 4. *Let $x \geq 10$. The number of positive integers $n \leq x$ such that $\tau(\sigma(n)) > \exp(\sqrt{\log x})$ is $O(x/(\log x)^2)$.*

Proof. Theorem 1 in [6] shows that

$$\sum_{n \leq x} \tau(\phi(n)) = x \exp\left(c(x) \left(\frac{\log x}{\log\log x}\right)^{1/2} \left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right),$$

where $c(x) \in [e^{-\gamma/2}/7, 2\sqrt{2}e^{-\gamma/2}]$ and $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. The remarks on page 128 of the same paper show that the above estimate holds with $\phi(n)$ replaced by $\sigma(n)$. In particular,

$$\begin{aligned} \#\{n \leq x : \tau(\sigma(n)) > \exp(\sqrt{\log x})\} \exp(\sqrt{\log x}) &\leq \sum_{n \leq x} \tau(\sigma(n)) \\ &< x \exp\left(O\left(\frac{\log x}{\log\log x}\right)^{1/2}\right), \end{aligned}$$

which gives that

$$\begin{aligned} \#\{n \leq x : \tau(\sigma(n)) > \exp(\sqrt{\log x})\} &< x \exp\left(-\sqrt{\log x} + O\left(\left(\frac{\log x}{\log\log x}\right)^{1/2}\right)\right) \\ &= O\left(\frac{x}{(\log x)^2}\right). \end{aligned}$$

□

2.2. **The Subspace Theorem and Linearly Recurrent Sequences.** Here, we review a quantitative version of the Subspace Theorem due to Evertse from [1] and apply it to nondegenerate linearly recurrent sequences of integers. Let $\mathbb{K}$ be an algebraic number field with ring of integers $\mathcal{O}_{\mathbb{K}}$ and collection of places (equivalence classes of absolute values) $M_{\mathbb{K}}$. For $v \in M_{\mathbb{K}}$ and $x \in \mathbb{K}$, we define the absolute value $|x|_v$ as follows:

$$|x|_v := \begin{cases} |\sigma(x)|^{\frac{1}{[\mathbb{K}:\mathbb{Q}]}}, & \text{if } v \text{ corresponds to } \sigma : \mathbb{K} \mapsto \mathbb{R}; \\ |\sigma(x)|^{\frac{2}{[\mathbb{K}:\mathbb{Q}]}}, & \text{if } v \text{ corresponds to the pair } \sigma, \overline{\sigma} : \mathbb{K} \mapsto \mathbb{C}; \\ N(\pi)^{-\frac{\mathrm{ord}_\pi(x)}{[\mathbb{K}:\mathbb{Q}]}}, & \text{if } v \text{ corresponds to the prime ideal } \pi \subset \mathcal{O}_{\mathbb{K}}. \end{cases}$$

Here, $N(\pi) := \#(\mathcal{O}_{\mathbb{K}}/\pi)$ is the norm of π and $\mathrm{ord}_\pi(x)$ is the exponent of π in the factorization of the principal fractional ideal (x) of $\mathbb{K}$ with the convention that $\mathrm{ord}_\pi(0) = \infty$. In the first two cases above, we call v real infinite or complex infinite, respectively, whereas in the third case we call v finite. These absolute values satisfy the product formula

$$\prod_{v\in M_{\mathbb{K}}} |x|_v = 1 \qquad \text{for all } x \in \mathbb{K}^*.$$

Now let $s \geq 2$, $\mathbf{x} := (x_1, \ldots, x_s) \in \mathbb{K}^s$ with $\mathbf{x} \neq 0$, and define

$$|\mathbf{x}|_v := \begin{cases} \left(\sum_{i=1}^s |x_i|_v^{2[\mathbb{K}:\mathbb{Q}]}\right)^{\frac{1}{2[\mathbb{K}:\mathbb{Q}]}}, & \text{if } v \text{ is real infinite;} \\ \left(\sum_{i=1}^s |x_i|_v^{[\mathbb{K}:\mathbb{Q}]}\right)^{\frac{1}{[\mathbb{K}:\mathbb{Q}]}}, & \text{if } v \text{ is complex infinite;} \\ \max\{|x_1|_v, \ldots, |x_s|_v\}, & \text{if } v \text{ is finite.} \end{cases}$$

Note that for infinite places v, $|\cdot|_v$ is a power of the Euclidean norm. Define

$$H(\mathbf{x}) := \prod_{v\in M_{\mathbb{K}}} |\mathbf{x}|_v.$$

In the statement of the next result, the following apply:

- $\mathbb{K}$ is an algebraic number field;
- $\mathcal{S}$ is a finite subset of $M_{\mathbb{K}}$ of cardinality r containing all the infinite places;
- $\{l_{1,v}, \ldots, l_{s,v}\}$ for $v \in \mathcal{S}$ are linearly independent sets of linear forms with algebraic coefficients in s variables such that

$$H(l_{i,v}) \leq H, \qquad [\mathbb{K}(l_{i,v}) : \mathbb{K}] \leq D$$

for all $i = 1, \ldots, s$ and $v \in \mathcal{S}$.

The following is the main Theorem in [1].

Theorem 2. *Let $0 < \delta < 1$. Consider the inequality*

$$\prod_{v\in\mathcal{S}} \prod_{i=1}^{s} \frac{|l_{i,v}(\mathbf{x})|_v}{|\mathbf{x}|_v} < \left(\prod_{v\in\mathcal{S}} |\det(l_{1,v}, \ldots, l_{s,v})|_v\right) H(\mathbf{x})^{-s-\delta}, \qquad \mathbf{x} \in \mathbb{K}^s. \tag{5}$$

Then

(i) *There are proper linear subspaces $T_1, \ldots, T_{t_1}$ with*

$$t_1 \leq (2^{60s^2}\delta^{-7s})^r \log(4D) \log\log(4D)$$

such that every solution $\mathbf{x} \in \mathbb{K}^s$ to (5) with $H(\mathbf{x}) \geq H$ satisfies

$$\mathbf{x} \in T_1 \cup \cdots \cup T_{t_1}.$$

(ii) *There are proper linear subspaces $S_1, S_2, \ldots, S_{t_2}$ of $\mathbb{K}^s$ with*

$$t_2 \leq (150s^4\delta^{-1})^{sr+1}(2 + \log\log(2H))$$

such that every solution $\mathbf{x} \in \mathbb{K}^s$ of (5) with $H(\mathbf{x}) < H$ satisfies

$$\mathbf{x} \in S_1 \cup S_2 \cup \cdots \cup S_{t_2}.$$

We present an application to small values of nondegenerate linearly recurrent sequences. But before, let us record the following result of Schmidt [8]. For a nondegenerate linearly recurrent sequence $(U_n)_{n\geq 1}$, let

$$\mathcal{Z}_U := \#\{n : U_n = 0\}.$$

Theorem 3. *If $(U_n)_{n\ge1}$ is a nondegenerate linearly recurrent sequence of order $k \ge 2$ whose terms are complex numbers, then*

$$\#\mathcal{Z}_U \le \exp(\exp(\exp(3k\log k))).$$

Now let $(U_n)_{n\ge1}$ be a nondegenerate linearly recurrent sequence of integers given by recurrence (1), whose characteristic polynomial is given by (3) and formula for the general term (2). Assume that $|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_s|$ and that $|U_n|$ is not a polynomial in n for large n. In particular, $|\alpha_1| > 1$.

We prove the following lemma.

Lemma 5. *Let $(U_n)_{n\ge1}$ be a nondegenerate linearly recurrent sequence of integers whose general term is given by (2) with $s \ge 2$ and assume that $|\alpha_1| = \max\{|\alpha_j| : 1 \le j \le s\}$. Then there exists x_0 and $c := c(U) \in (0, 1/3)$ such that for $x \ge x_0$, the number of $n \le x$ such that*

$$|U_n| \le |\alpha_1|^{n(1-\delta)}, \tag{6}$$

with $\delta := x^{-c}$ is of cardinality $O_U(\sqrt{x})$.

Proof. We may assume that $n \in (x^{1/2}, x]$ since there are only $O(x^{1/2})$ positive integers $n \le x^{1/2}$. Using (2), inequality (6) becomes

$$\left|\sum_{i=1}^{s} P_i(n)\alpha_i^n\right| \le |\alpha_1|^{n(1-\delta)}.$$

Let L be a positive integer that is a common multiple of all the denominators of all the coefficients of $P_i(X)$ for $i = 1, \ldots, s$. Multiplying across by L, we get, by setting $Q_i(X) := LP_i(X)$, that

$$\left|\sum_{i=1}^{s} Q_i(n)\alpha_i^n\right| \le L|\alpha_1|^{n(1-\delta)}. \tag{7}$$

Note now that $Q_i(n)\alpha_i^n \in \mathcal{O}_{\mathbb{K}}$, where $\mathbb{K} := \mathbb{Q}(\alpha_1, \ldots, \alpha_s)$ is an algebraic number field. For technical reasons, we would like to exclude the greatest common divisor of the ideals $(\alpha_1), \ldots, (\alpha_s)$. So, let $I := \gcd((\alpha_1), \ldots, (\alpha_s))$. Then I^h is principal for some positive integer h, which can be taken to be the cardinality of the class group of $\mathbb{K}$. Let β be a generator of I^h. Then β divides α_i^h for all $i = 1, \ldots, s$, so $(\alpha_1^h/\beta), \ldots, (\alpha_s^h/\beta)$ are coprime. Since I is Galois invariant, any conjugate $\beta^{(j)}$ of β is also a generator of I, so β is associated to any of its conjugates. Letting d be the degree of β, we get that $\alpha_i^h/\beta^{(j)}$ are all associated for $j = 1, \ldots, d$ (and fixed $i \in \{1, \ldots, s\}$) and in particular, they are also associated with α_i^{hd}/b, where we can take $b := N(\beta)$. Now we replace $(U_n)_{n\ge1}$ with any of the hd linearly recurrent sequences $(U_{hdm+\ell})_{m\ge0}$ and $\ell \in \{0, 1, \ldots, hd-1\}$ by fixing ℓ. Then

$$U_{hdm+\ell} = b^m \sum_{i=1}^{s} Q_i'(m)\alpha_i'^m,$$

where $Q_i'(X) := \alpha_i^\ell Q_i(hdX + \ell) \in \mathcal{O}_{\mathbb{K}}[X]$ and $\alpha_i' := \alpha_i^{hd}/b$ for $i = 1, \ldots, s$. Inequality (7) now implies

$$\left|\sum_{i=1}^{s} Q_i'(m)\alpha_i'^m\right| \le L|\alpha_1|^\ell \left|\frac{\alpha_1^{hd}}{b}\right|^m \cdot \alpha_1^{-\delta(hdm+\ell)} = L'|\alpha_1'|^{m(1-\delta_1)}, \tag{8}$$

where $L' := L|\alpha_1|^{\ell(1-\delta)}$ and $\delta_1 := c_0\delta$ with $c_0 := \dfrac{hd\log|\alpha_1|}{hd\log|\alpha_1| - \log b}$. Note that $|\alpha_1'| > 1$, for if not, then $|\alpha_i^{hd}/b| \le 1$ holds for all $i = 1, \ldots, s$. Since α_i' are algebraic integers having all the conjugates at most 1, we get that they are roots of unity. Thus, α_i/α_j is a root of unity for all $i \neq j$, which contradicts the nondegeneracy assumption.

We now set up the subspace machinery. We let $\mathcal{S}$ be the subset of $M_{\mathbb{K}}$ containing all the infinite valuations as well as all the finite ones v such that $|\alpha_i'|_v \neq 1$ for some $i = 1, \ldots, s$. We take $\mathbf{x} = (x_1, \ldots, x_s)$ and $l_{i,v}(\mathbf{x})$ given by

$$l_{i,v}(\mathbf{x}) := x_i \qquad \text{for all } (i, v) \in \{1, \ldots, s\} \times \mathcal{S} \text{ with } i \geq 2 \text{ or } v \text{ finite},$$

and take

$$l_{1,v}(\mathbf{x}) := x_1 + \cdots + x_s \qquad \text{for } v \text{ infinite}.$$

We evaluate

$$\prod_{i=1}^{s} \prod_{v \in \mathcal{S}} |l_{i,v}(\mathbf{x})|_v \tag{9}$$

in $\mathbf{x} := (Q_1'(m)\alpha_1'^m, \ldots, Q_s'(m)\alpha_s'^m)$ with some m satisfying inequality (8). For a fixed $i \geq 2$, we have

$$\prod_{v \in \mathcal{S}} |l_{i,v}(\mathbf{x})|_v = \prod_{v \in \mathcal{S}} |Q_i'(m)\alpha_i'^m|_v \leq \prod_{v \text{ infinite}} |Q_i'(m)|_v \ll m^{\sigma_i - 1},$$

where the implied constant depends on the coefficients of $Q_i'(x)$. The above inequality follows by the product formula for $\alpha_i'^m$, together with the fact that $\mathcal{S}$ contains all the places of $M_{\mathbb{K}}$ for which $|\alpha_i'|_v \neq 1$ and together with the fact that $Q_i'(m)$ is an algebraic integer, so $|Q_i'(m)|_v \leq 1$ for all finite places v. Hence,

$$\prod_{i=2}^{s} \prod_{v \in \mathcal{S}} |l_{i,v}(\mathbf{x})|_v \ll \prod_{i=2}^{s} m^{\sigma_i - 1} = m^{\sum_{i=2}^{s} \sigma_i - 1}.$$

For $i = 1$, we have that

$$\begin{aligned} \prod_{v \in \mathcal{S}} |l_{1,v}(\mathbf{x})|_v &= \prod_{\substack{v \in \mathcal{S} \\ v \text{ finite}}} |Q_1'(m)\alpha_1'^m|_v \prod_{\substack{v \in \mathcal{S} \\ v \text{ infinite}}} |\sum_{i=1}^{s} Q_i'(m)\alpha_i'^m|_v \\ &\ll \prod_{\substack{v \in \mathcal{S} \\ v \text{ infinite}}} |Q_i'(m)|_v \left(\prod_{\substack{v \in \mathcal{S} \\ v \text{ finite}}} |\alpha_1'^m|_v \right) |\alpha_1'|^{m(1-\delta_1)}. \end{aligned}$$

In the above, we used the fact that $\sum_{i=1}^{s} Q_i'(m)\alpha_i'^m$ is an integer from $\mathbb{Z}$, so the product of its valuations over all infinite places $v \in M_{\mathbb{K}}$ is just the regular absolute value of this integer. Using again the product formula, $|\alpha_1'^m|$ is cancelled by the second product above, so we get that

$$\prod_{v \in \mathcal{S}} |l_{1,v}(\mathbf{x})|_v \ll m^{\sigma_1 - 1}(|\alpha_1'^m|)^{-\delta_1}.$$

Collecting everything together, we get that the product shown in (9) is bounded as

$$\prod_{i=1}^{s} \prod_{v \in \mathcal{S}} |l_{i,v}(\mathbf{x})|_v \ll m^{\sum_{i=1}^{s} \sigma_i - 1}(|\alpha_1'^m|)^{-\delta_1} \ll m^k |\alpha_1'^m|^{-\delta_1}. \tag{10}$$

To be able to apply Theorem 2, we should compare the above upper bound on our double product with

$$\left(\prod_{v \in \mathcal{S}} |\det(l_{1,v}, l_{2,v}, \ldots, l_{s,v})|_v \right) \left(\frac{\prod_{v \in \mathcal{S}} |\mathbf{x}|_v}{H(\mathbf{x})} \right)^s H(\mathbf{x})^{-\delta_2}$$

for some suitable δ_2. Well, the first factor above is easy since all the involved determinants are equal to 1. For the second factor above, we have that

$$\begin{aligned}\frac{\prod_{v\in\mathcal{S}}|\mathbf{x}|_v}{H(\mathbf{x})} &= \prod_{v\in M_{\mathbb{K}}\backslash\mathcal{S}}|\mathbf{x}|_v^{-1} = \prod_{v\in M_{\mathbb{K}}\backslash\mathcal{S}}(\max\{|Q_i'(m)\alpha_i'^m|_v\})^{-1}\\ &= \prod_{v\in M_{\mathbb{K}}\backslash\mathcal{S}}(\max\{|Q_i'(m)|_v\})^{-1}\geq 1.\end{aligned}$$

In the above, we used the fact that $M_{\mathbb{K}}\backslash\mathcal{S}$ consists only of finite valuations v for which $|\alpha_i'^m|_v=1$. Finally for $H(\mathbf{x})$, we use the fact that $x_i\in\mathcal{O}_{\mathbb{K}}$ to deduce that

$$H(\mathbf{x})\leq\prod_{v\ \text{infinite}}|\mathbf{x}|_v\leq\left(\sum_{i=1}^{s}|Q_i'(m)^2\alpha_i'|^{2m}\right)^{1/2}\ll m^k|\alpha_1'|^m.$$

Here, we used the fact that $\sum_{i=1}^{s}Q_i'(m)^2\alpha_i'^{2m}$ is an integer as the collection of numbers $\{Q_1'(m)^2\alpha_1'^{2m},\ldots,Q_s'(m)\alpha_s'^{2m}\}$ is Galois stable. In particular, we have that

$$\left(\prod_{v\in\mathcal{S}}|\det(l_{1,v},l_{2,v},\ldots,l_{s,v})|_v\right)\left(\frac{\prod_{v\in\mathcal{S}}|\mathbf{x}|_v}{H(\mathbf{x})}\right)^s H(\mathbf{x})^{-\delta_2}\gg m^{-k\delta_2}|\alpha_1'^m|^{-\delta_2}.$$

So, inequality (5) will hold for us, assuming that

$$c_1m^k|\alpha_1'^m|^{-\delta_1}\leq c_2m^{-k\delta_2}|\alpha_1'^m|^{-\delta_2} \tag{11}$$

holds, where c_1 and c_2 are the constants implied by the $\ll$ and $\gg$ symbols in (10) and (11), respectively. We take $\delta_2:=\delta_1/2$ and the above inequality becomes equivalent to

$$(c_1/c_2)m^{k(1+\delta_2)}<|\alpha_1'^m|^{\delta_1/2}.$$

Taking logarithms, we get

$$k(1+\delta_2)\log m+\log(c_1/c_2)\leq(\delta_1/2)(\log|\alpha_1'|)m.$$

Since $n\leq x$, then the left side is $O(\log x)$. Since $\delta_1=c_0\delta\gg x^{-1/3}$, $\log|\alpha_1'|>0$, and $m\gg n\gg\sqrt{x}$, it follows that the right side above is $\gg x^{1/6}$. Thus, the last inequality above holds for $x>x_0$, where x_0 depends on U. We conclude that our $\mathbf{x}$ satisfies inequality (5) with $\delta_2:=\delta_1/2$ and $x>x_0$.

We take a closer look at $H(\mathbf{x})$. Since $(\alpha_1'),\ldots,(\alpha_s')$ are coprime, it follows that for every finite place $v\in M_{\mathbb{K}}$, there is $i\in\{1,\ldots,s\}$ such that $|\alpha_i'|_v=1$. This shows that for finite v, we have

$$|\mathbf{x}|_v\gg\min\{|Q_i'(m)|_v\}\gg m^{-k}.$$

Hence,

$$H(\mathbf{x})\gg m^{-rk}\prod_{v\ \text{infinite}}|Q_i'(m)\alpha_i'^m|_v\gg m^{-rk}|\alpha_1'|^m. \tag{12}$$

Here, r is the cardinality of $\mathcal{S}$. For our set-up, the parameter H can be taken to be $\sqrt{s}$. Since $m\gg n\gg x^{1/2}$, it follows that for large x, the inequality

$$c_3m^{-rk}|\alpha_1'|^m\geq\sqrt{s}$$

holds, where c_3 is the constant implied in (12). Thus, for $x>x_0$, we have

$$H(\mathbf{x})\geq H.$$

Also, for us $D = 1$, since $l_{i,v}(\mathbf{x})$ have coefficients from $\mathbb{Z}$. So, by Theorem 2, there are proper subspaces $T_1, \ldots, T_{t_1}$ with

$$t_1 \le (2^{60s^2} \delta_2^{-7s})^r \log 4 \log\log 4$$

such that $\mathbf{x} \in T_1 \cup T_2 \cup \cdots \cup T_{t_1}$. Each of the containments $\mathbf{x} \in T_j$ leads to an equation of the form

$$\sum_{i=1}^{s} C_i^{(j)} Q_i'(m) \alpha_i'(m) = 0,$$

where $\mathbf{C}^{(j)} := (C_1^{(j)}, \ldots, C_s^{(j)}) \in \mathbb{K}^s$ is not the zero vector. Each such equation signals that m is in the set of zeros of a nondegenerate linearly recurrent sequence of order at most k, so there are at most $O_k(1)$ such values of m, where the constant in O_k can be taken to be $\exp(\exp(\exp(3k \log k)))$ by Theorem 3. So, it remains to understand the upper bound on t_2. But, this is

$$(2^{60s^2+7s} c_0^{-7s} x^{7sc})^r \log 4 \log\log 4.$$

Taking $c := 1/(15sr)$, the above bound becomes

$$2^{(60s^2+7s)r} c_0^{-7sr} x^{7/15} \log 4 \log\log 4,$$

and this is smaller than $x^{1/2}$ for $x > x_0$. This finishes the proof. □

3. Proof of Theorem 1

3.1. **The Case of the Euler ϕ Function.** Let $p(n)$ be the smallest prime factor of n. Let

$$\mathcal{A}_1(x) = \{n \le x : p(n) > x^{c_1}\},$$

where $c_1 \in (0, 1/6)$ is a constant to be determined later. We show first that $\mathcal{A}_1(x)$ contains $O(x/\log x)$ positive integers $n \le x$ as $x \to \infty$. Indeed, putting $y := x^{c_1}$, the set $\mathcal{A}_1(x)$ coincides with the $n \le x$, which are coprime to all the primes $p \le y$. The number of such is, by Lemma 2,

$$\Phi(x, y) \ll \frac{x}{\log y} \ll \frac{x}{\log x}.$$

From now on, let $n \le x$ not in $\mathcal{A}_1(x)$. We also assume that $n \ge x^{1/2}$, since there are only $O(x^{1/2}) = o(x/\log x)$ as $x \to \infty$ positive integers failing this last inequality.

For such n, the interval $[1, n]$ contains at least $n/p(n)$ numbers, which are not coprime to n, namely all the positive integers that are multiples of $p(n)$. Thus,

$$\phi(n) \le n - \frac{n}{p(n)} \le n - n\delta,$$

where $\delta := 1/x^{c_1}$. Let

$$U_n = \sum_{i=1}^{s} P_i(n) \alpha_i^n.$$

We assume that $|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_s|$. Assume first that $s = 1$. In this case $\Psi(X) = (X - \alpha_1)^k$, so α_1 is an integer with $|\alpha_1| \ge 2$. Thus,

$$U_n = P_1(n) \alpha_1^n,$$

where $P_1(X) \in \mathbb{Q}[X]$. Let L be the least common denominator of all the coefficients of $P_1(X)$. Then, for large n (say larger than the maximal real root of $P_1(X)$), we have

$$\begin{aligned} L\phi(|U_n|) &\geq \phi(L|U_n|) \geq \phi(L|P_1(n)|)\phi(|\alpha_1|^n) \gg \frac{L|P_1(n)|}{\log\log(L|P_1(n)|)}|\alpha_1|^n \\ &\gg \frac{Ln^{k-1}}{\log\log n}|\alpha_1|^n. \end{aligned}$$

This gives

$$\phi(|U_n|) \gg \frac{n^{k-1}}{\log\log n}|\alpha_1|^n. \tag{13}$$

On the other hand,

$$|U_{\phi(n)}| = |P_1(\phi(n))||\alpha_1|^{\phi(n)} \ll \phi(n)^{k-1}|\alpha_1|^{n(1-\delta)} \ll n^{k-1}|\alpha_1|^{n(1-\delta)}. \tag{14}$$

By (13) and (14), it follows that if

$$\phi(|U_n|) \leq |U_{\phi(n)}|$$

holds, then

$$\frac{n^{k-1}}{\log\log n}|\alpha_1|^n \leq \phi(|U_n|) \leq |U_{\phi(n)}| \ll n^{k-1}|\alpha_1|^{n(1-\delta)}.$$

This is equivalent to

$$|\alpha_1|^{n\delta} \ll \log\log n.$$

Taking logarithms, this becomes

$$(n\delta)\log|\alpha_1| \leq \log\log\log n + O(1).$$

The right side is $O(\log\log\log x)$. Since $|\alpha_1| \geq 2$, $n\delta \geq x^{1/2}/x^{c_1} \geq x^{1/3}$; it follows that the above inequality implies that x is bounded. Thus, there are only finitely many such n in case $s = 1$.

From now on, we assume $s \geq 2$. In this case, the inequality

$$|\alpha_1|^{n/2} \leq |U_n| \ll |\alpha_1|^n n^k$$

holds for all $n \geq n_0$. Indeed, the right side is obvious and the left side follows from a known application of the Subspace Theorem (see, for example, [11, Lemma 4.1]). Thus, for $n \geq n_0$, we have that

$$\log\log|U_n| = \log n + O(1).$$

Since $\phi(m) \gg m/\log\log m$ holds for all integers $m \geq 2$ (see Lemma 1), we have that for $n > n_0$, the inequality

$$\phi(|U_n|) \gg \frac{|U_n|}{\log\log|U_n|} = \frac{|U_n|}{\log n + O(1)}$$

holds. Assume now that

$$\phi(|U_n|) \leq |U_{\phi(n)}|.$$

We then get

$$\frac{|U_n|}{\log n + O(1)} \ll \phi(|U_n|) \leq |U_{\phi(n)}| \leq |\alpha_1|^{\phi(n)}\phi(n)^k \ll |\alpha_1|^{n-n\delta}n^k.$$

This gives

$$|U_n| \ll |\alpha_1|^{n-n\delta}n^k(\log n + O(1)).$$

Let c_4 be the constant implied by the above inequality and c_5 be the constant implied by the above $O(1)$. We claim that with $\delta_1 := 1/x^{2c_1}$, the inequality

$$c_4|\alpha_1|^{n(1-\delta)}n^k(\log n + c_5) < |\alpha_1|^{n(1-\delta_1)}$$

holds. Indeed, this is equivalent to

$$k\log n + \log(\log n + c_5) + \log(c_4) < n(\delta - \delta_1)\log|\alpha_1|. \tag{15}$$

Since $n \in (\sqrt{x}, x]$, the left side above is $O(\log x)$. Since $\delta_1 = 1/x^{2c_1}$ and $\delta = 1/x^{c_1}$, it follows that $\delta - \delta_1 \geq 0.5\delta \geq 0.5x^{-c_1}$ for $x \geq x_0$. Since $n \geq \sqrt{x}$, it follows that the right side above is $\gg x^{1/2-c_1} \gg x^{1/3}$. Therefore, indeed (15) holds for all our n in $(\sqrt{x}, x]$ and $x > x_0$. Thus, we get

$$|U_n| \leq |\alpha_1|^{n(1-\delta_1)}.$$

By Lemma 5, we can choose $c_1 := c/2$ such that the number of $n \leq x$ satisfying the above inequality is $O_k(\sqrt{x}) = o(x/\log x)$ as $x \to \infty$, which finishes the argument.

3.2. **The Case of the σ Function.** Assume again that x is large and $n \in (\sqrt{x}, x]$ is divisible by some prime $p \leq x^{c_1}$ for some small constant c_1, since otherwise, like in the case of the Euler function, the set of such $n \leq x$ is $O(x/\log x)$. Then $\sigma(n) \geq n + n\delta$, where $\delta := 1/x^{c_1}$, which gives

$$n \leq \frac{\sigma(n)}{1+\delta} \leq \sigma(n)(1-\delta_1),$$

where $\delta_1 := \delta/2$. Assume now that

$$|U_{\sigma(n)}| \leq \sigma(|U_n|). \tag{16}$$

As in the case of the ϕ function, we need to treat the case $s = 1$ separately. In this case, $U_n = P_1(n)\alpha_1^n$, where $|\alpha_1| \geq 2$ is an integer and $P_1(X) \in \mathbb{Q}[X]$. Let again L be the least common denominator of the coefficients of $P_1(X)$ and n be larger than the maximal real zero of $P_1(X)$. The right side above is by Lemma 1.

$$\begin{aligned}\sigma(|U_n|) \leq \sigma(L|U_n|) &= \sigma(L|P_1(n)||\alpha_1|^n) \leq \sigma(L|P_1(n)|)\sigma(|\alpha_1|^n)\\ &\ll L|P_1(n)|(\log\log(L|P_1(n)|))|\alpha_1|^n\\ &\ll n^{k-1}(\log\log n)|\alpha_1|^{\sigma(n)(1-\delta_1)}, \end{aligned} \tag{17}$$

whereas

$$|U_{\sigma(n)}| = |P_1(\sigma(n))||\alpha_1|^{\sigma(n)} \gg \sigma(n)^{k-1}|\alpha_1|^{\sigma(n)} \gg n^{k-1}|\alpha_1|^{\sigma(n)}. \tag{18}$$

Inequality (16), with (17) and (18), imply

$$n^{k-1}|\alpha_1|^{\sigma(n)} \ll |U_{\sigma(n)}| \leq \sigma(|U_n|) \ll n^{k-1}(\log\log n)|\alpha_1|^{\sigma(n)(1-\delta_1)}.$$

This leads to

$$|\alpha_1|^{\delta_1\sigma(n)} \ll \log\log n,$$

and by the argument for the case $s = 1$ and the ϕ function, this leads to the conclusion that x (so, n) is bounded.

From now on, we assume that $s \geq 2$. The right side is, by Lemma 1 and the calculation done at the case of the Euler ϕ function,

$$\begin{aligned}\sigma(|U_n|) &\ll |U_n|\log\log|U_n| \ll |\alpha_1|^n n^k(\log n + O(1))\\ &\leq |\alpha_1|^{\sigma(n)(1-\delta_1)}\sigma(n)^k(\log\sigma(n) + O(1)).\end{aligned}$$

Let c_6 and c_7 be the constants implied by the $\ll$-symbol and O-symbol above, respectively. By the argument done in the case of the Euler ϕ function, putting $\delta_2 := 1/x^{2c_1}$, the inequality

$$c_6|\alpha_1|^{m(1-\delta_1)}m^k(\log m + c_7) < |\alpha_1|^{m(1-\delta_2)}$$

holds for all $m = \sigma(n)$ and $n \in (\sqrt{x}, x]$ for $x > x_0$. Thus, putting $m := \sigma(n)$, we get that

$$|U_m| \leq |\alpha_1|^{m(1-\delta_2)}$$

holds, when $x \geq x_0$. Note that $m \ll x \log\log x$ by Lemma 1. By Lemma 5, we can choose $c_1 = c/2$ and then the set of $m \ll x \log\log x$ satisfying the above inequality is of cardinality

$$O_k(\sqrt{x \log\log x}).$$

But this is only an upper bound on the number of distinct values of $\sigma(n)$ and we have to get an upper bound on the number of n's themselves. By Lemmas 3 and 4, we may assume that $\Omega(n) \leq 10 \log\log x$ and $\tau(\sigma(n)) \leq \exp(\sqrt{\log x})$, since the number of $n \leq x$ for which one of the above inequalities fails is $O(x/(\log x)^2)$. Writing

$$n = p_1^{a_1} \cdots p_\ell^{a_\ell},$$

with distinct primes $p_1, \ldots, p_\ell$ and positive exponents $a_1, \ldots, a_\ell$, we have

$$\sigma(n) = \prod_{i=1}^{\ell} \left(\frac{p_i^{a_i+1} - 1}{p_i - 1} \right).$$

Given $m = \sigma(n)$, each of $(p_i^{a_i+1} - 1)/(p_i - 1)$ is a divisor d_i of $\sigma(n)$. Additionally, given d_i and also a_i, p_i is uniquely determined. Thus, since d_i can be fixed in at most $\tau(\sigma(n))$ ways and $a_i \leq \Omega(n)$ can be fixed in at most $\Omega(n)$ ways, it follows that $p_i^{a_i}$ can be fixed in at most $\Omega(n)\tau(\sigma(n))$ ways. This is so for a fixed i, but $i \leq \ell = \omega(n) \leq \Omega(n)$. Thus, the number of such n, when $\sigma(n)$ and $\Omega(n)$ are given, is at most

$$\Big((10 \log\log x) \exp(\sqrt{\log x})\Big)^{10 \log\log x} < \exp\Big(20(\log\log x)\sqrt{\log x}\Big)$$

for $x > x_0$. Varying $\Omega(n)$ up to $10 \log\log x$, as well as the number of possible values of $\sigma(n)$, we get that the number of possible $n \leq x$ is

$$\ll_k \sqrt{x \log\log x}(\log\log x) \exp\Big(20(\log\log x)\sqrt{\log x}\Big) = o(x/\log x)$$

as $x \to \infty$, which finishes the proof of the σ case.

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