ON CERTAIN FIBONACCI REPRESENTATIONS

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Abstract. One of the most popular and studied recursive sequences is the Fibonacci sequence. It is challenging to see how Fibonacci numbers can be used to generate other recursive sequences. In our article, we describe some families of integer recurrence sequences as rational polynomial linear combinations of Fibonacci numbers.

1. Introduction

As usual, the sequence of Fibonacci numbers $(F_n)_{n=0}^{\infty}$ is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ (see sequence A000045 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [4]). Terms with negative subscripts $-n$ ($n \in \mathbb{N}$) can be introduced via the equality

$$F_{-n} = -F_{-n+1} + F_{-n+2}, \tag{1.1}$$

and it turns out that $F_{-n} = (-1)^{n+1}F_n$. The zeros of the characteristic polynomial $x^2 - x - 1$ are $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. The Binet formula gives the Fibonacci numbers explicitly by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$.

Several problems of combinatorics have solutions in the form

$$w_n := u(n)F_n + v(n)F_{n-1} + c(n), \tag{1.2}$$

where $u(x)$, $v(x)$, and $c(x)$ are rational polynomials of the variable x. It is not obvious, at least at the beginning, that the terms of w_n in (1.2) are integers, because the coefficient polynomials are rational.

For example, if a_n gives the number of parts in all compositions of $n + 1$ with no 1s, then

$$a_n = \frac{2n+3}{5}F_n - \frac{n}{5}F_{n-1},$$

see A010049 [4]. Here, $u(x) = (2x + 3)/5$ and $v(x) = -x/5$ are linear polynomials with noninteger rational coefficients (and $c(x)$ vanishes), but (a_n) is an integer sequence. There are polynomials with higher degree appearing in (1.2). For instance, consider the sequence $a_n =$ A129707 of [4], which describes the number of inversions in all Fibonacci binary words of length n. The formula

$$a_{n-3} = \frac{5n^2 - 37n + 50}{50}F_n + \frac{4n-4}{50}F_{n-1} \tag{1.3}$$

given in the encyclopedia leads to

$$a_n = \frac{5n^2 - n - 4}{25}F_n + \frac{5n^2 + n}{50}F_{n-1}$$

via the identity $z_3F_{n+3} + z_2F_{n+2} = (3z_3 + 2z_2)F_n + (2z_3 + z_2)F_{n-1}$. The last identity comes immediately when one applies the Fibonacci recurrence thrice. An extension having a similar

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flavor is based on the well-known Fibonacci identity $F_{n-j} = F_nF_{-j} + F_{n-1}F_{-j+1}$ with $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Combining it with (1.1), we have

$$F_{n-j} = \left((-1)^j F_{j-1}\right) F_n + \left((-1)^{j+1} F_j\right) F_{n-1}. \tag{1.4}$$

In this paper, we study the following general problem. Let $0 \le j_1 < j_2 < \cdots < j_s$ be nonnegative integers, and $p_1(x), p_2(x), \ldots, p_s(x) \in \mathbb{Q}[x]$ such that $\deg(p_i(x)) = d_i$. Put $d^\star = \max_i\{d_i\}$, which is a nonnegative integer. Define the sequence $(w_n)_{n\in\mathbb{Z}}$ by

$$w_n = p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s}. \tag{1.5}$$

The following question is the main subject of the present work. Can we give conditions for the rational functions $p_i(x)$ to guarantee that the sequence (w_n) is integral? Later, we will answer the question for certain families of polynomials.

Remark 1. *Multiple applications of (1.4) transforms (1.5) into the form*

$$w_n = P_0(n)F_n + P_1(n)F_{n-1}, \tag{1.6}$$

where $P_0(x)$ and $P_1(x)$ are suitable rational polynomials depending on the subscripts $j_1, j_2, \ldots, j_s$ and on the polynomials $p_1(x), p_2(x), \ldots, p_s(x)$. Hence, essentially, it is sufficient to consider only (1.6). But, sometimes the form (1.5) promises a more advantageous starting point in the investigations.

In general, a linear recurrence with constant coefficients is called a C-finite sequence, where the character C refers to the constant coefficients. It is known that the set of C-finite sequences is closed under the operations of addition and multiplication (see [2, Chapter 4]).

The question arises naturally whether integer sequences (w_n) having type (1.6) (or equivalently (1.5)) are C-finite. And are they closed under addition?

For the first question, consider (1.6). This admits

$$w_n = P_0(n)F_n + P_1(n)F_{n-1} = \frac{1}{\sqrt{5}}\left(P_0(n) + \frac{P_1(n)}{\alpha}\right)\alpha^n + \frac{1}{\sqrt{5}}\left(P_0(n) + \frac{P_1(n)}{\beta}\right)\beta^n.$$

The last expression shows that the corresponding characteristic polynomial of the sequence (w_n) has zeros α and β with certain multiplicities (which can be derived from the coefficient polynomials, respectively). Hence, the constant coefficients of the characteristic polynomial provide the coefficients of the linear recurrence for (w_n). Thus, (w_n) is C-finite.

The closure property (for addition) of sequences (w_n) having type (1.6) can be easily seen. Indeed, taking two such sequences (w_n) and $(w_n^\star)$, their sum is given by

$$w_n + w_n^\star = (P_0(n)F_n + P_1(n)F_{n-1}) + (P_0^\star(n)F_n + P_1^\star(n)F_{n-1}) = P_0^{\star\star}(n)F_n + P_1^{\star\star}(n)F_{n-1},$$

where $P_0^{\star\star}(n) = P_0(n) + P_0^\star(n)$ and $P_1^{\star\star}(n) = P_1(n) + P_1^\star(n)$. The integrity of the sum sequence is obvious because (w_n) and $(w_n^\star)$ are integer sequences.

In Section 2, after considering the general case (1.5), we examine only (1.6) for rational coefficient polynomials with small degree. In Section 3, a modified version of (1.2) will be studied.

2. Main Results

2.1. **General Approach.** Although Remark 1 of the previous section provides the idea how to simplify (1.5) to get (1.6); here, we choose a slightly different way. At the beginning, we assume that $p_j(x) \in \mathbb{C}[x]$ for $j = 1, 2, \ldots, s$.

The Binet formula implies that

$$\begin{aligned} w_n &= p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s} \\ &= \sum_{t=1}^{s} p_t(n)\frac{\alpha^{n-j_t} - \beta^{n-j_t}}{\sqrt{5}} \\ &= \sum_{t=1}^{s}\left(\frac{p_t(n)}{\alpha^{j_t}}\frac{\alpha^n}{\sqrt{5}} - \frac{p_t(n)}{\beta^{j_t}}\frac{\beta^n}{\sqrt{5}}\right). \end{aligned} \tag{2.1}$$

Then, there exist polynomials $q_\alpha(x), q_\beta(x) \in \mathbb{C}[x]$ (if $p_j(x) \in \mathbb{Q}[x]$, then $q_\alpha(x)$ and $q_\beta(x)$ are from $\mathbb{Q}(\alpha)[x]$) such that

$$w_n = q_\alpha(n)\alpha^n - q_\beta(n)\beta^n. \tag{2.2}$$

Clearly,

$$q_\alpha(n) = \sum_{t=1}^{s}\frac{p_t(n)}{\sqrt{5}\alpha^{j_t}}, \quad q_\beta(n) = \sum_{t=1}^{s}\frac{p_t(n)}{\sqrt{5}\beta^{j_t}}.$$

Let $d_\alpha = \deg(q_\alpha(x))$ and $d_\beta = \deg(q_\beta(x))$. Put $\tilde{d} = d_\alpha + d_\beta + 2$, which gives the order of the recursive sequence (w_n). The characteristic polynomial of (w_n) is

$$c_w(x) = (x-\alpha)^{d_\alpha+1}(x-\beta)^{d_\beta+1} = (x^2-x-1)^{d_{\alpha\beta}+1}(x-\alpha)^{d_\alpha-d_{\alpha\beta}}(x-\beta)^{d_\beta-d_{\alpha\beta}},$$

where $d_{\alpha\beta} = \min\{d_\alpha, d_\beta\}$. Note that at least one of $d_\alpha - d_{\alpha\beta}$ and $d_\beta - d_{\alpha\beta}$ is zero.

Before investigating the principal problem, we analyze the question of equality of degrees d_α and d_β. In the case $s = 2$, $j_1 = 0$, $j_2 = 1$, the example

$$w_n = (n+1)F_n + (-\alpha n + 2)F_{n-1},$$

where $p_1(n) = n+1$ and $p_2(n) = -\alpha n + 2$ admits $q_\alpha(n) = 1$ and $q_\beta(n) = \alpha n - 1$, so it might happen that d_α differs from d_β. In the example above, the coefficients are not from $\mathbb{Q}$, but from $\mathbb{Q}(\sqrt{5})$, and this is the reason why $d_\alpha \neq d_\beta$ may happen. The situation differs when we assume $p_j(x) \in \mathbb{Q}[x]$ for all possible j. In this case, one can easily show that $d_\alpha = d_\beta$. Here, we skip the proof because it is rather technical, but we note the crucial point. The leading coefficients of $q_\alpha(x)$ and $q_\beta(x)$ are conjugates in $\mathbb{Q}(\sqrt{5})$, so they can vanish only together.

2.2. **Specific Cases with Equal Degrees.** In the sequel, suppose that the coefficient polynomials are from $\mathbb{Q}$, i.e., $d_\alpha = d_\beta$. Consequently $d_{\alpha\beta} = d_\alpha = d_\beta$, and then $\tilde{d} = 2(d_{\alpha\beta}+1)$ holds; moreover

$$c_w(x) = (x^2-x-1)^{d_{\alpha\beta}+1}.$$

We note in advance that the method we use in Cases 2.2.1–2.2.3 (and essentially in Section 3) can be applied for other given coefficient polynomials $p_j(x)$. We always obtain a system of parametric linear equations, where the unknowns are the coefficients of the polynomials $p_j(x)$ and their multipliers come from the initial values of (w_n). The evaluation of the solution leads to the desired conditions.

2.2.1. *Case $s = 2$, $j_1 = 0$, $j_2 = 1$, $d_1 = d_2 = 1$.* Assume that $a \neq 0$, b, $c \neq 0$, and d are rational numbers and

$$w_n = (an+b)F_n + (cn+d)F_{n-1}. \tag{2.3}$$

Following the list of equivalent transformations in (2.1) leads to

$$w_n = \frac{(a\alpha+c)n + (b\alpha+d)}{\alpha\sqrt{5}}\alpha^n - \frac{(a\beta+c)n + (b\beta+d)}{\beta\sqrt{5}}\beta^n; \tag{2.4}$$

the initial values are

$$\begin{aligned} w_0 &= d, \\ w_1 &= a+b, \\ w_2 &= 2a+b+2c+d, \\ w_3 &= 6a+2b+3c+d. \end{aligned}$$

The characteristic polynomial of (w_n) is

$$c_w(x) = (x-\alpha)^2(x-\beta)^2 = (x^2-x-1)^2 = x^4-2x^3-x^2+2x+1;$$

hence, the recurrence relation

$$w_n = 2w_{n-1} + w_{n-2} - 2w_{n-3} - w_{n-4} \tag{2.5}$$

holds for $n \geq 4$.

Now, we investigate what rational coefficients a, b, c, and d guarantee that (w_n) is integral. Clearly, the initial values w_0, w_1, w_2, and w_3 must be integers. Consequently, $d = w_0$ must be an integer, further solving the system

$$\begin{aligned} z_1 &= a+b, \\ z_2 &= 2a+b+2c, \\ z_3 &= 6a+2b+3c \end{aligned}$$

in a, b, and c with arbitrary integer parameters $z_1 = w_1$, $z_2 = w_2 - d$, and $z_3 = w_3 - d$, we obtain

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5}.$$

This result, together with (2.5), guarantees that (w_n) is an integer sequence. Hence, we have proved:

Theorem 2.1. *The terms*

$$w_n = (an+b)F_n + (cn+d)F_{n-1}$$

form an integer sequence (w_n) *if and only if* d *is an integer and*

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5},$$

where z_1*,* z_2*, and* z_3 *are integers, too.*

Note that once we have $d \in \mathbb{Z}$ and $a, b, c \in \mathbb{Q}$, as given in the theorem above, then the initial values of the recurrence (w_n) of order four are $w_0 = d$, $w_1 = z_1$, $w_2 = z_2 + d$, and $w_3 = z_3 + d$. Thus, the integer sequence (2.5) with suitable initial values has another interpretation given by (2.3). For example, let $d = 0$; also, $z_1 = z_2 = 1$ and $z_3 = 3$. In this case, we get the integer sequence

$$w_n = \frac{2n+3}{5}F_n - \frac{n}{5}F_{n-1},$$

which is the first example in the introduction.

2.2.2. *Case $s = 2$, $j_1 = 0$, $j_2 = 1$, $d_1 = d_2 = 2$.* This subsection is devoted to the study of the case when the coefficient polynomials are quadratic. The treatment is analogous to the previous subsection; hence, we give only the results of computations.

Assume that $a \neq 0$, b, c, $d \neq 0$, e, and f are rational numbers, and

$$w_n = (an^2 + bn + c)F_n + (dn^2 + en + f)F_{n-1}. \tag{2.6}$$

Now, sequence (w_n) satisfies

$$w_n = \frac{(a\alpha + d)n^2 + (b\alpha + e)n + (c\alpha + f)}{\alpha\sqrt{5}}\alpha^n - \frac{(a\beta + d)n^2 + (b\beta + e)n + (c\beta + f)}{\beta\sqrt{5}}\beta^n, \tag{2.7}$$

with initial values

$$\begin{aligned} w_0 &= f, \\ w_1 &= a + b + c, \\ w_2 &= 4a + 2b + c + 4d + 2e + f, \\ w_3 &= 18a + 6b + 2c + 9d + 3e + f, \\ w_4 &= 48a + 12b + 3c + 32d + 8e + 2f, \\ w_5 &= 125a + 25b + 5c + 75d + 15e + 3f. \end{aligned} \tag{2.8}$$

The characteristic polynomial of (w_n) is

$$c_w(x) = (x - \alpha)^3(x - \beta)^3 = (x^2 - x - 1)^3 = x^6 - 3x^5 + 5x^3 - 3x - 1;$$

hence,

$$w_n = 3w_{n-1} - 5w_{n-3} + 3w_{n-5} + w_{n-6}. \tag{2.9}$$

Clearly, f must be an integer, further eliminating f from system (2.8) and solving it in a, b, c, d, and e, we obtain

$$\begin{aligned} a &= \frac{-z_1 + 3z_2 + z_3 - 3z_4 + z_5}{10}, \\ b &= \frac{-5z_1 - 75z_2 + 15z_3 + 45z_4 - 17z_5}{50}, \\ c &= \frac{30z_1 + 30z_2 - 10z_3 - 15z_4 + 6z_5}{25}, \\ d &= \frac{3z_1 - 4z_2 - 3z_3 + 4z_4 - z_5}{10}, \\ e &= \frac{-45z_1 + 80z_2 + 15z_3 - 40z_4 + 11z_5}{50}, \end{aligned} \tag{2.10}$$

where $z_1 = w_1$, $z_2 = w_2 - f$, $z_3 = w_3 - f$, $z_4 = w_4 - 2f$, and $z_5 = w_5 - 3f$ are arbitrary integer parameters. A summary of the result of this subsection is:

Theorem 2.2. *The rational coefficients a, b, c, d, e, and f determine integers sequences of the form*

$$w_n = (an^2 + bn + c)F_n + (dn^2 + en + f)F_{n-1}$$

if and only if $f \in \mathbb{Z}$ and a, b, c, d, and e are as given in (2.10).

Thus, the integer sequences (2.9) with suitable initial values also have another interpretation given by (2.6). For example, let $f = 0$, $z_1 = 0$, $z_2 = 1$, $z_3 = 4$, $z_4 = 12$, and $z_5 = 31$. In this particular case, we get the integer sequence

$$w_n = \frac{5n^2 - n - 4}{25}F_n + \frac{5n^2 + n}{50}F_{n-1},$$

which is equivalent to the result (1.3) given in OEIS [4].

2.2.3. *A Particular Case with Nonequal Degrees:* $s = 2$, $j_1 = 0$, $j_2 = 1$, $d_1 = 2$, $d_2 = 1$. Now, $a \neq 0$, b, c, $d \neq 0$, and e are all in $\mathbb{Q}$, and

$$w_n = (an^2 + bn + c)F_n + (dn + e)F_{n-1}. \tag{2.11}$$

Using the usual technique, we obtain that sequence (w_n) satisfies

$$w_n = \frac{a\alpha n^2 + (b\alpha + d)n + (c\alpha + e)}{\alpha\sqrt{5}}\alpha^n - \frac{a\beta n^2 + (b\beta + d)n + (c\beta + e)}{\beta\sqrt{5}}\beta^n, \tag{2.12}$$

with initial values

$$\begin{aligned} w_0 &= e, \\ w_1 &= a + b + c, \\ w_2 &= 4a + 2b + c + 2d + e, \\ w_3 &= 18a + 6b + 2c + 3d + e, \\ w_4 &= 48a + 12b + 3c + 8d + 2e. \end{aligned} \tag{2.13}$$

The characteristic polynomial of (w_n) is

$$c_w(x) = (x - \alpha)^3(x - \beta)^3 = (x^2 - x - 1)^3 = x^6 - 3x^5 + 5x^3 - 3x - 1;$$

hence,

$$w_n = 3w_{n-1} - 5w_{n-3} + 3w_{n-5} + w_{n-6}. \tag{2.14}$$

Clearly, e must be an integer; further eliminating e from (2.13) and solving it in a, b, c, and d, we obtain

$$\begin{aligned} a &= \frac{2z_1 - z_2 - 2z_3 + z_4}{10}, \\ b &= \frac{-56z_1 - 7z_2 + 66z_3 - 23z_4}{50}, \\ c &= \frac{48z_1 + 6z_2 - 28z_3 + 9z_4}{25}, \\ d &= \frac{-6z_1 + 18z_2 - 9z_3 + 2z_4}{25}, \end{aligned} \tag{2.15}$$

where $z_1 = w_1$, $z_2 = w_2 - e$, $z_3 = w_3 - e$, and $z_4 = w_4 - 2e$ are arbitrary integer parameters. A summary of the result of this subsection is:

Theorem 2.3. *The rational coefficients a, b, c, d, and e determine integer sequences of the form*

$$w_n = (an^2 + bn + c)F_n + (dn + e)F_{n-1}$$

if and only if e and z_i $(i = 1, \ldots, 4)$ are integers and a, b, c, and d are as given in (2.15).

For example, let $e = z_1 = z_2 = z_3 = 1$ and $z_4 = 2$. In this particular case, we get the integer sequence

$$w_n = \frac{5n^2 - 43n + 88}{50}F_n + \frac{14n + 50}{50}F_{n-1}.$$

This sequence $(w_n)_0^\infty = (1, 1, 2, 2, 4, 7, 15, 32, 69, 146, 303, \ldots)$ does not appear in OEIS.

3. A Modified Problem

In the introduction, (1.2) offers a further polynomial $c(x)$. Németh [3] investigated a related question, namely the problem of walks on tiled square boards, and proved, among others things, that the tiling-walking sequence (r_n) of the $(2\times n)$-board with only dominoes is recursively given by a sixth order recurrence having explicit form

$$r_n = \frac{4n}{5}F_{n+1} + \frac{3n+3}{5}F_n + \frac{1}{2} + \frac{1}{2}(-1)^n. \tag{3.1}$$

This is sequence A054454 in [4].

Our purpose now is to examine the sequence

$$w_n = (an+b)F_n + (cn+d)F_{n-1} + e + f(-1)^n, \tag{3.2}$$

where the coefficients $a, b, \dots, f$ are rational numbers, again, to have integrity condition for (w_n).

Because the method is detailed in the previous parts of Section 2, we record the statement, and compare it to Németh's equality (3.1).

Theorem 3.1. *Let the initial values $w_0, w_1, \dots, w_5$ be integers. If*

$$a = \frac{3w_0+2w_1-7w_2-w_3+4w_4-w_5}{5}, \qquad b = \frac{-3w_0-2w_1-3w_2+6w_3+6w_4-4w_5}{5},$$

$$c = \frac{-4w_0-w_1+11w_2-2w_3-7w_4-3w_5}{5}, \quad d = 2w_1+w_2+2w_3-w_4,$$

$$e = \frac{w_0+3w_1+w_2-3w_3-w_4+w_5}{2}, \qquad f = \frac{w_0+w_1-3w_2-w_3+3w_4-w_5}{2},$$

then $w_n = (an+b)F_n + (cn+d)F_{n-1} + e + f(-1)^n$ is an integer sequence. The converse of this statement is also true.

As an example, let $w_0 = 0$, $w_1 = 1$, $w_2 = 2$, $w_3 = 6$, $w_4 = 12$, and $w_5 = 26$. Now, $a = 4/5$, $b = -4/5$, $c = 3/5$, $d = 0$, $e = 1/2$, and $f = -1/2$. Then

$$w_n = \frac{4n-4}{5}F_n + \frac{3n}{5}F_{n-1} + \frac{1}{2} - \frac{1}{2}(-1)^n.$$

This coincides with (3.1) via $r_n = w_{n+1}$.

Finally, we give a well-known sequence for $f = 0$ in (3.2). The sequence of Leonardo numbers is defined by $L_0 = 1$, $L_1 = 1$, and by $L_n = L_{n-1} + L_{n-2} + 1$ (cited as A001595 in OEIS [4]). It is easy to see that

$$L_n = 2F_n + 2F_{n-1} - 1.$$

Recently, Atanassov [1] defined some recurrence sequences as linear combinations of two consecutive Fibonacci numbers. Moreover, the reader will find more special examples in its references.

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