GENERATING SOLUTIONS FOR A SPECIAL CLASS OF DIOPHANTINE EQUATIONS II

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In [1] I gave a method for generating solutions for a special class of Diophantine equations. In this note I will give a method for generating solutions for another special class of Diophantine equations.

In [2] Carmichael gives a method for generating solutions for the Diophantine equation:

$$kx^3 + ax^2y + bxy^2 + cy^3 = t^2 \tag{1}$$

If one examines equation (1) carefully one will note that $kx^3 + ax^2y + bxy^2 + cy^3$ is a homogeneous polynomial of degree 3 and that t^2 has degree 2. Note that 3 and 2 are relatively prime. In this note we shall show how to generate solutions for a class of Diophantine equations of which (1) is a special case.

We begin with $p(x_1, x_2, \ldots, x_k)$ a polynomial with positive integer coefficients, homogeneous of degree n, where n is an integer, and $n > 1$. Let b be a positive integer and m an integer, $m > 1$. Suppose that $(m, n) = 1$. I.e. m and n are relatively prime. The class of Diophantive equations we consider is given by:

$$p(x_1, x_2, \ldots, x_k) = bt^m. \tag{2}$$

Let $(a_1, a_2, \ldots, a_k)$ be a k-tuple of positive integers and $p(a_1, a_2, \ldots, a_k) = a$. There are three possibilities:

(i) $a = bc^m$ for some positive integer c,

(ii) $a = bc$, i.e. a is divisible by b,

(iii) $a = c$, i.e. b does not divide a.

In case (i) we were lucky to guess a solution of (2).

In case (ii) there exists positive integers e_1 and e_2 such that $me_1 = ne_2 + 1$. (Remember that $(m, n) = 1$).

Now $c^{ne_2}a = bcc^{ne_2} = bc^{me_1}$.

Hence $(c^{e_2})^n(p(a_1, a_2, \ldots, a_k) = b(c^{e_1})^m$.

Therefore $p(c^{e_2}a_1, c^{e_2}a_2, \ldots, c^{e_2}a_k) = b(c^{e_1})^m$ or $x_i = c^{e_2}a_i, i = 1$ to k and

$$t = c^{e_i}$$

is a solution of (2).

The following example illustrates the method described above.

Example: Starting with the equation

$$2x^3 + x^2y + 2xy^2 + y^3 = 3t^2$$

we substitute $x = 1$ and $y = 1$ to obtain $a = 6$. In this case $b = 3$ and $c = 2$. Further $m = 2$ and $n = 3$. Now $2 \cdot 2 = 3 \cdot 1 + 1$. Hence $e_1 = 2$ and $e_2 = 1$. Therefore $(2^1)^3[2(1)^3 + (1)^2(1) + 2(1)(1)^2 + (1)^3] = 3 \cdot 2 \cdot 2^3 = 3(2^2)^2$. Hence $2(2)^3 + (2)^2(2) + 2(2)(2)^2 + (2)^3 = 3(4)^2$.

Our solution is $x = 2, y = 2$ and $t = 4$.

In case (iii) there exists positive integers d_1, d_2, e_1 and e_2 such that $me_1 = ne_2 + 1$ and $nd_2 = md_1 + 1$.

Now $b^{nd_2}c^{ne_2}a = b^{nd_2}c^{ne_2}c$.

Hence $(b^{d_2}c^{e_2})^n p(a_1, a_2, \dots, a_k) = b(b^{d_1}c^{e_1})^m$.

Therefore $p(b^{d_2}c^{e_2}a_1, b^{d_2}c^{e_2}a_2, \dots, b^{d_2}c^{e_2}a_k) = b(b^{d_1}c^{e_1})^m$ or $x_i = b^{d_2}c^{e_2}a_i, i = 1$ to k and

$$t = b^{d_1}c^{e_1}$$

is a solution of (2).

The following example illustrates the method described above.

Example: Starting with the equation

$$2x^3 + x^2y + 2xy^2 + y^3 = 3t^2$$

we substitute $x = 1$ and $y = 2$ to obtain $a = 20$. In this case $c = 20$. Further $m = 2$ and $n = 3$. Now, as before, $2 \cdot 2 = 3 \cdot 1 + 1$. Hence $e_1 = 2$ and $e_2 = 1$. Further $1 \cdot 3 = 1 \cdot 2 + 1$. Hence $d_2 = 1$ and $d_1 = 1$. Therefore $(3^1 \cdot 20^1)^3[2(1)^3 + (1)^2(2) + 2(1)(2)^2 + (2)^3] = 3(3^1 \cdot 20^2)^2$. Hence $2(3\cdot 20)^3 + (3\cdot 20)^2(2\cdot 3\cdot 20) + 2(3\cdot 20)(2\cdot 3\cdot 20)^2 + (2\cdot 3\cdot 20)^3 = 3(3\cdot 20^2)^2$. Our solution is $x = 60, y = 120$ and $t = 1200$.

Of course in applying this method we can fine tune it depending on the value of a. For example if in case (ii) $a = bc = bd^n c_1$ we need only multiply by $c_1^{ne_2}$. And similarly for case (iii) if $b = b_1b_2$ and $c = b_1c_1$, i.e. $(b, c) = b_1$, we need only multiply by $b_2^{nd_2}$ and then proceed as in case (ii).

Finally, if m is odd, we can extend the class of equations to $p(x_1, x_2, \dots, x_k)$, a polynomial with non-zero integer coefficents, b a non-zero integer, and $(a_1, a_2, \dots, a_k)$ a k-tuple of non-zero integers, with all other conditions remaining the same.

REFERENCES

[1] P.J. Arpaia. "Generating Solutions For a Special Class of Diophantine Equations." *The Fibonacci Quarterly* **32.2** (1994): 170-173.

[2] R.D. Carmichael. *Diophantine Analysis.* N.Y. John Wiley and Sons, 1915.

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