A NOTE ON SUMMATION OF CERTAIN RECIPROCAL SERIES INVOLVING THE GENERALIZED FIBONACCI AND LUCAS FUNCTIONS

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1. INTRODUCTION

Consider the Fibonacci and Lucas functions $F(x)$ and $L(x)$ introduced by

$$F(x) = \frac{\mu^x - e^{i\pi x}\mu^{-x}}{\sqrt{5}} \tag{1}$$

and

$$L(x) = \mu^x + e^{i\pi x}\mu^{-x}, \tag{2}$$

respectively, where x is a real variable and $\mu = (1+\sqrt{5})/2$, see [5]. In the following, we assume that $x > 0$. It is clear that, for a nonnegative integer $n, F(n)$ and $L(n)$ are the n^{th} terms of the well-known Fibonacci and Lucas sequences, which play an important role in many subjects such as algebra, geometry, and number theory itself.

For the summation of reciprocal series involving Fibonacci and Lucas numbers, it is difficult to compute. Up to now, one cannot find an effective method. Recently, there are a number of publications dealing with this kind of work, see [2-4, 6-10], and the summation of reciprocal series related to the Fibonacci and Lucas functions, see [1, 11]. In this paper, we consider the generalized Fibonacci and Lucas functions given by

$$W(x) = \frac{A\alpha^x - Be^{i\pi x}\alpha^{-x}}{\Delta^{1/2}}, \tag{3}$$

where A and B are two constants and

$$\alpha = \frac{p+\sqrt{\Delta}}{2}, \quad \Delta = p^2 + 4,$$

with $p > 0$. We will establish some identities involving reciprocals of the products of these generalized functions.

2. MAIN RESULTS

Note that, from the definition of the function $W(x)$,

$$W(x) = \begin{cases} F(x), & A = B = p = 1 \\ L(x), & A = -B = \Delta^{1/2} \text{ and } p = 1. \end{cases}$$

In what follows, when $A = B = 1$, we write $W(x) = U(x)$. Then, the main conclusion can be stated as follows.

Theorem: Suppose that m is a positive integer. Then we have

$$\sum_{n=1}^{m} \frac{\alpha^{nx} e^{in\pi x}}{W(nx)W(nx+x)W(nx+2x)} = \frac{X_1 + X_2 - X_{m+1} - X_{m+2}}{A^2 U(x)U(2x)} - \frac{X_2 - X_{m+2}}{A^2 \alpha^x U^2(x)}, \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{\alpha^{nx} e^{in\pi x}}{W(nx)W(nx+x)W(nx+2x)} = \frac{X_1 + X_2}{A^2 U(x)U(2x)} - \frac{X_2}{A^2 \alpha^x U^2(x)}, \tag{5}$$

$$\sum_{n=1}^{m} \frac{\alpha^{2nx} e^{in\pi x}}{W(nx)W(nx+x)W(nx+2x)W(nx+3x)}$$

$$= \frac{X_1 + X_2 + X_3 - X_{m+1} - X_{m+2} - X_{m+3}}{A^3 U(x)U(2x)U(3x)}$$

$$+ \frac{X_3 - X_{m+3}}{A^3 \alpha^{3x} U^3(x)} - \frac{2(X_3 - X_{m+3}) + X_2 - X_{m+2}}{A^3 \alpha^{2x} U^2(x) U(2x)}, \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{\alpha^{2nx} e^{in\pi x}}{W(nx)W(nx+x)W(nx+2x)W(nx+3x)}$$

$$= \frac{X_1 + X_2 + X_3}{A^3 U(x)U(2x)U(3x)} + \frac{X_3}{A^3 \alpha^{3x} U^3(x)} - \frac{2X_3 + X_2}{A^3 \alpha^{2x} U^2(x) U(2x)}, \tag{7}$$

where $X_n = \frac{e^{in\pi x}}{\alpha^{nx} W(nx)}$.

Proof: We only demonstrate the proof of identities (4) and (5). In a similar way, the other two equalities can be proved. In fact, from(3), we have

$$\frac{e^{in\pi x}}{W(nx)W(nx+kx)} = \frac{1}{AU(kx)} \left(\frac{e^{in\pi x}}{\alpha^{nx} W(nx)} - \frac{e^{i(n+k)\pi x}}{\alpha^{(n+k)x} W(nx+kx)} \right). \tag{8}$$

It then follows that

$$\frac{\alpha^{nx} e^{in\pi x}}{W(nx)W(nx+x)W(nx+2x)}$$

$$= \frac{1}{A^2 U(x)U(2x)} \left(\frac{e^{in\pi x}}{\alpha^{nx} W(nx)} - \frac{e^{i(n+2)\pi x}}{\alpha^{(n+2)x} W(nx+2x)} \right)$$

$$- \frac{1}{A^2 \alpha^x U^2(x)} \left(\frac{e^{i(n+1)\pi x}}{\alpha^{(n+1)x} W(nx+x)} - \frac{e^{i(n+2)\pi x}}{\alpha^{(n+2)x} W(nx+2x)} \right). \tag{9}$$

Summing on both sides of (9), we obtain identity (4). On the other hand, since $\lim_{m\to\infty} X_m = 0$, there holds equality (5). This completes the proof. □

According to the particular choices of A, B, and p in (4-7), we can work out some identities related to $F(x)$ and $L(x)$. For example, in (4),

(I) If $A = B = p = 1$, we have

$$\sum_{n=1}^{m} \frac{\mu^{nx} e^{in\pi x}}{F(nx)F(nx+x)F(nx+2x)} = \frac{1}{F(x)F(2x)} \left(\frac{e^{i\pi x}}{\mu^{x} F(x)} + \frac{e^{2i\pi x}}{\mu^{2x} F(2x)} - \frac{e^{i(m+1)\pi x}}{\mu^{(m+1)x} F(mx+x)} - \frac{e^{i(m+2)\pi x}}{\mu^{mx+2x} F(mx+2x)} \right) - \frac{1}{\mu^{x} F^2(x)} \left(\frac{e^{2i\pi x}}{\mu^{2x} F(2x)} - \frac{e^{i(m+2)\pi x}}{\mu^{mx+2x} F(mx+2x)} \right);$$

(II) if $A = -B = \Delta^{1/2}$ and $p = 1$, we obtain

$$\sum_{n=1}^{m} \frac{\mu^{nx} e^{in\pi x}}{L(nx)L(nx+x)L(nx+2x)} = \frac{1}{5F(x)F(2x)} \left(\frac{e^{i\pi x}}{\mu^{x} L(x)} + \frac{e^{2i\pi x}}{\mu^{2x} L(2x)} - \frac{e^{i(m+1)\pi x}}{\mu^{mx+x} L(mx+x)} - \frac{e^{i(m+2)\pi x}}{\mu^{mx+2x} L(mx+2x)} \right) - \frac{1}{5\mu^{x} F^2(x)} \left(\frac{e^{2i\pi x}}{\mu^{2x} L(2x)} - \frac{e^{i(m+2)\pi x}}{\mu^{mx+2x} L(mx+2x)} \right).$$

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REFERENCES

[1] R. André-Jeannin. "Generalized Complex Fibonacci and Lucas Functions." *The Fibonacci Quarterly* **29.1** (1991): 13-18.

[2] R. André-Jeannin. "Summation of Certain Reciprocal Series Related to Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **29.3** (1991): 200-204.

[3] R. Backstrom. "On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression." *The Fibonacci Quarterly* **19.1** (1981): 14-21.

[4] Brother Alfred Brousseau. "Summation of Infinite Fibonacci Series." *The Fibonacci Quarterly* **7.2** (1969): 143-168

[5] A.F. Horadam & A.G. Shannon. "Fibonacci and Lucas Curves." *The Fibonacci Quarterly* **26.1** (1988): 3-13.

[6] R.S. Melham & A.G. Shannon. "On Reciprocal Sums of Chebyshev Related Sequences." *The Fibonacci Quarterly* **33.3** (1995): 194-202.

[7] R.S. Melham. "Reduction Formulas for the Summing of Reciprocals in Certain Second-Order Recurring Sequences." *The Fibonacci Quarterly* **40.1** (2002): 71-75.

[8] B. Popov. "On Certain Series of Reciprocals of Fibonacci Numbers." *The Fibonacci Quarterly* **20.3** (1982): 261-265.

[9] B. Popov. "Summation of Reciprocal Series of Numerical Functions of Second Order." *The Fibonacci Quarterly* **24.1** (1986): 17-21.

[10] Feng-Zhen Zhao "Notes on Reciprocal Series Related to Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **37.3** (1999): 254-257.

[11] Feng-Zhen Zhao & Tianming Wang. "Some Identities for the Generalized Fibonacci and Lucas Functions." *The Fibonacci Quarterly* **39.5** (2001): 436-438.

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