ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by March 15, 2005. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-981 Proposed by Steve Edwards, Southern Polytechnic State University Marietta, GA

For all positive integers n, prove that

$$\sum_{k=1}^{n} F_{6k} = F_{6n+5} - \frac{5}{4}F_{3n+3}^2 + \frac{(-1)^n - 1}{2}.$$

B-982 Proposed by Harris Kwong, SUNY Fredonia, Fredonia NY

For odd positive integers k, evaluate the sums

$$\sum_{n=2}^{\infty} \frac{F_{kn}}{F_{k(n-1)}F_{k(n+1)}} \text{ and } \sum_{n=2}^{\infty} \frac{L_{kn}}{L_{k(n-1)}L_{k(n+1)}}.$$

B-983 Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain

Let n be a nonnegative integer. Prove that the system of equations

$$\frac{1}{2}(x+y+z) = F_{n+2},$$

$$\frac{1}{2}(x^2+y^2+z^2) = F_{n+2}^2 - F_nF_{n+1},$$

$$\frac{1}{2}(x^3+y^3+z^3) = F_{n+2}^3 - \frac{3}{2}F_nF_{n+1}F_{n+2}$$

has only integer solutions and determine them.

B-984 Proposed by Juan Pla, Paris, France

Find solutions related to Lucas and Fibonacci numbers to the Diophantine equation

$$x^2 - 5y^2 + 2z^2 = \pm 1.$$

B-985 Proposed by Mario Catalani, University of Torino, Torino, Italy

Let $P_0 = 0, P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Define $U_n = F_{p_n}$ and $V_n = L_{p_n}$. For $n \geq 0$, show that

(a) $U_{n+2} = \frac{1}{2}\left(U_n(V_{n+1}^2 - 2(-1)^{n+1}) + U_{n+1}V_{n+1}\sqrt{5U_n^2 + 4(-1)^n}\right)$ with $U_0 = 0$ and $U_1 = 1$;

(b) $V_{n+2} = \frac{1}{2}\left(V_n(V_{n+1}^2 - 2(-1)^{n+1}) + U_{n+1}V_{n+1}\sqrt{5V_n^2 - 20(-1)^n}\right)$ with $V_0 = 2$ and $V_1 = 1$.

SOLUTIONS

A Recurrence Relation

B-966 Proposed by Stanley Rabinowitz, Math Pro, Westford, MA (Vol. 41, no. 5, Nov. 2003)

Find a recurrence relation for $r_n = \frac{1}{1+F_n}$.

Solution by Steve Edwards, Southern Polytechnic State U., Marietta, GA

$$r_{n+1} = \frac{1}{1+F_{n+1}} = \frac{1}{1+F_n+F_{n-1}} = \frac{1}{(1+F_n)+(1+F_{n-1})-1} = \frac{1}{\frac{1}{r_n}+\frac{1}{r_{n-1}}-1},$$

which can also be written $\frac{r_n r_{n-1}}{r_n + r_{n-1} - r_n r_{n-1}}$.

Also solved by Brian Beasley, José Luis Diaz-Barrero, Ovidiu Furdui, Pentti Haukkanen, Russell Hendel, Gerald A. Heuer, H.-J. Seiffert, James Sellers, and the proposer.

A Fibonacci Integral Pattern

B-967 Proposed by Juan Pla, Paris, France (Vol. 41, no. 5, Nov. 2003)

Prove that $\frac{5}{32}F_{6n}^2$ is an integer of the form $\frac{m(m+1)}{2}$.

Solution by H.-J. Seiffert, Berlin, Germany

Since $2 = F_3$ divides F_{3n},

$$m = \begin{cases} \frac{5}{4}F_{3n}^2 - 1 & \text{if } n \text{ is odd} \\ \frac{5}{4}F_{3n}^2 & \text{if } n \text{ is even} \end{cases}$$

is an integer. From (I_{12}) and (I_7) of [1], we have $5F_{3n}^2 + 4(-1)^n = L_{3n}^2$ and $F_{3n}L_{3n} = F_{6n}$. It follows that $\frac{m(m+1)}{2} = \frac{5}{32}F_{6n}^2$.

Reference

1. V.E. Hoggatt, Jr. Fibonacci and Lucas Numbers. Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Brian Beasley, Steve Edwards, Ovidiu Furdui, James Sellers, Harris Kwong, and the proposer.

Find its Limit!

B-968 Proposed by Mohammad K. Azarian, University of Evansville, Evansville IN
(Vol. 41, no. 5, Nov. 2003)

Let $F(n) = \sum_{i=2}^{n} \frac{4+1000F_i}{F_{i-1}F_{i+1}}$ where F_i is the i^{th} Fibonacci number. Find $\lim_{n\to\infty} F(n)$.

Solution by Pentti Haukkanen, University of Tampere, Tampere, Finland

It is known that

$$\sum_{i=2}^{\infty} \frac{1}{F_{i-1}F_{i+1}} = 1,$$

and

$$\sum_{i=2}^{\infty} \frac{F_i}{F_{i-1}F_{i+1}} = 2,$$

see e.g. [1], Exercise 35, p. 442, and Exercise 25, p. 441. This shows that

$$\lim_{n\to\infty} F(n) = 2004.$$

Reference:

1. Thomas Koshy. Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics. Wiley-Interscience, New York, 2001.

Most other solvers actually verified the values of the two sums referred to in the featured solutions.

Also solved by Charles Cook, Kenneth Davenport, José Luis Diaz-Barrero, Steve Edwards, Ovidiu Furdui, Russell Hendel, Harris Kwong, H.-J. Seiffert, and the proposer.

Much Ado About 4/3

B-969 Proposed by José Luis Díaz-Barrero, UPC, Barcelona, Spain
(Vol. 41, no. 5, Nov. 2003)

Evaluate the following sum

$$\sum_{n=1}^{\infty} \frac{F_{n+1}[F_{2n+3} + (-1)^{n+1}]F_{n+3}}{F_{n+2}[F_{2n+1} + (-1)^{n}][F_{2n+5} + (-1)^{n+2}]}.$$

A Composite Solution by Ovidiu Furdui, Western Michigan University, and H.-J. Seiffert, Berlin, Germany

¿From (I_{21}) and (I_{23}) of [1], we know that, for all positive integers $n, F_{2n+1} + (-1)^n = F_{n+1}L_n, F_{2n+3} + (-1)^{n+1} = F_{n+2}L_{n+1}$, and $F_{2n+5} + (-1)^{n+2} = F_{n+3}L_{n+2}$. Now notice that

$$\frac{F_{n+1}[F_{2n+3} + (-1)^{n+1}]F_{n+3}}{F_{n+2}[F_{2n+1} + (-1)^n][F_{2n+5} + (-1)^{n+2}]} = \frac{F_{n+1} \cdot F_{n+2}L_{n+1} \cdot F_{n+3}}{F_{n+2} \cdot F_{n+1}L_n \cdot L_{n+2}F_{n+3}} = \frac{L_{n+1}}{L_nL_{n+2}} = \frac{1}{L_n} - \frac{1}{L_{n+2}}.$$

Let $S_n = \sum_{k=1}^n \left(\frac{1}{L_k} - \frac{1}{L_{k+2}}\right) = \frac{1}{L_1} + \frac{1}{L_2} - \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}$. Therefore $S_n \to \frac{1}{L_1} + \frac{1}{L_2} = 1 + \frac{1}{3} = \frac{4}{3}$ as $n \to \infty$. It follows that the desired sum converges to $\frac{4}{3}$.

Also solved by Steve Edwards (Similar solution to the one above), Harris Kwong, and the proposer.

Three Formulas

B-970 Proposed by Peter G. Anderson, Rochester Institute of Technology Rochester, NY
(Vol. 41, no. 5, Nov. 2003)

Define a second-order and three third-order recursions by:

$$f_n = f_{n-1} + f_{n-2}, \text{ with } f_0 = 1, f_1 = 1.$$
$$g_n = g_{n-1} + g_{n-3}, \text{ with } g_0 = 1, g_1 = 1, g_2 = 1.$$
$$h_n = h_{n-2} + h_{n-3}, \text{ with } h_0 = 1, h_1 = 0, h_2 = 1.$$

and

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \text{ with } t_0 = 1, t_1 = 1, t_2 = 2.$$

Prove:

1. $t_{n+3} = f_{n+3} + \sum_{p+q=n} f_pt_q$.
2. $t_{n+2} = g_{n+2} + \sum_{p+q=n} g_pt_q$.
3. $t_{n+1} = h_{n+1} + \sum_{p+q=n} h_pt_q$.

Solution by John F. Morrison, Baltimore, MD

The formulas can be proved by induction, but the sums in the formulas suggest using generating functions. We first note that if $s_n = as_{n-1} + bs_{n-2} + cs_{n-3}$ then the generating function for the s_n is

$$S(x) = \sum_{m=0}^{\infty} s_mx^m = \frac{x^2(x_2 - as_1 - bs_0) + x(s_1 - as_0) + s_0}{1 - ax - bx^2 - cx^3}. \qquad (*)$$

To show this multiply out $\left(\sum_{m=0}^{\infty} s_mx^m\right)(1 - ax - bx^2 - cx^3)$ and note that all the terms after the one containing x^2 are zero because of the recurrence.

Then, substituting the coefficients and the initial values in (*), we have

$$F(x) = \sum_{m=0}^{\infty} f_m x^m = \frac{1}{1-x-x^2}$$

$$G(x) = \sum_{m=0}^{\infty} g_m x^m = \frac{1}{1-x-x^3}$$

$$H(x) = \sum_{m=0}^{\infty} h_m x^m = \frac{1}{1-x^2-x^3} \qquad (**)$$

$$T(x) = \sum_{m=0}^{\infty} t_m x^m = \frac{1}{1-x-x^2-x^3}$$

Since $\sum_{p+q=n} s_p t_q$ is the coefficient of x^n in $S(x)T(x)$ the formulas we wish to prove are equivalent to

(1) $T(x) = F(x) + x^3 F(x)T(x)$.
(2) $T(x) = G(x) + x^2 G(x)T(x)$.
(3) $T(x) = H(x) + xH(x)T(x)$.

But, from (**), we see

$$\frac{1}{F(x)} = \frac{1}{T(x)} + x^3$$

$$\frac{1}{G(x)} = \frac{1}{T(x)} + x^2$$

$$\frac{1}{H(x)} = \frac{1}{T(x)} + x$$

multiply these by $F(x)T(x), G(x)T(x), H(x)T(x)$, respectively, to get the desired equalities.

It should be noted that similar equalities can be found, in the same way, for any recurrences, which differ in one term if the initial values are chosen so that the munerator in (*) is 1,
i.e. if

$$\frac{1}{U(x)} = \frac{1}{V(x)} + kx^m$$

then

$$V(x) = U(x) + kx^m U(x)V(x)$$

and

$$v_{n+m} = u_{n+m} + k \sum_{p+q=n} u_p v_q.$$

Also solved by Harris Kwong, H.-J. Seiffert and the proposer.