SOME RESULTS ON GENERALIZED FIBONACCI AND LUCAS NUMBERS AND DEDEKIND SUMS

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ABSTRACT

In this paper, by using the reciprocity formula of Dedekind sums and the properties of generalized Fibonacci and Lucas numbers, the authors investigate Dedekind sums for generalized Fibonacci and Lucas numbers and generalize some conclusions of other authors.

1. INTRODUCTION

Recently, Dedekind sums for Fibonacci numbers were investigated and some meaningful results were obtained (see [7]). In this paper, the authors will consider Dedekind sums for generalized Fibonacci and Lucas numbers.

The Binet forms of generalized Fibonacci and Lucas numbers are:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $n \geq 0, \alpha = (p+\sqrt{\Delta})/2, \beta = (p-\sqrt{\Delta})/2, \Delta = p^2 - 4q > 0$, and p and q are integers with $pq \neq 0$. Throughout this paper, we assume that $p > 0$. When $n < 0$, we define $U_n = (-1)^n U_{-n}$ and $V_n = (-1)^n V_{-n}$. It is well known that $\{U_n\}$ and $\{V_n\}$ satisfy the recurrence relation

$$W_{n+2} = pW_{n+1} - qW_n, \quad n \geq 0. \tag{1}$$

For $p = -q = 1, \{U_n\}$ and $\{V_n\}$ are the classical Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$, respectively.

The Dedekind sum $S(h,t)$ is defined by

$$S(h,t) = \sum_{a=1}^{t} \left(\left(\frac{a}{t}\right)\right)\left(\left(\frac{ah}{t}\right)\right), \tag{2}$$

where t is a positive integer, h is an arbitrary integer, and

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

For various arithmetical properties of $S(h,t)$, several articles have been written (see [1], [4], and [6]). Regarding Dedekind sums and uniform distribution, Myerson [5] and Zheng [8] have obtained some meaningful results. In [7], Zhang studied the distribution problem of Dedekind sums for Fibonacci numbers F_n and obtained some interesting results. Zhang discussed the mean value distribution of $S(F_n, F_{n+1})$ and presented a sharper asymptotic formula for $\sum_{n=1}^{m} S(F_n, F_{n+1})$. Inspired by Zhang's results, we decided to investigate Dedekind sums for U_n and V_n. The principal purpose of this paper is to compute the values of $S(U_n, U_{n+1})$ and $S(V_n, V_{n+1})$ for $|q| = 1$, by the reciprocity formula of Dedekind sums and properties of $\{U_n\}$

and $\{V_n\}$. In the meantime, we present the sharper asymptotic formulas of $\sum_{n=1}^{m} S(U_n, U_{n+1})$ and $\sum_{n=1}^{m} S(V_n, V_{n+1})$ when $|q| = 1$.

2. MAIN RESULTS

In this section, we state and prove the main results of this paper.

Theorem 1: Assume that $q = -1$ and m is a positive integer. Then

$$S(U_{2m}, U_{2m+1}) = 0, \tag{3}$$

$$S(U_{2m+1}, U_{2m+2}) = -\frac{U_{2m}}{6pU_{2m+2}} + \frac{(p-1)(p-2)}{12p}, \tag{4}$$

$$S(V_{2m}, V_{2m+1}) = -\frac{U_{2m} + V_{2m-1}}{12pV_{2m+1}} + \frac{(p-1)(p-5)}{24p} - \frac{1}{12p}, \quad (2 \nmid p), \tag{5}$$

$$\begin{aligned} S(V_{2m+1}, V_{2m+2}) = & \frac{U_{2m}}{12pV_{2m+1}} + \frac{V_{2m+1}}{12V_{2m+2}} + \frac{1}{12V_{2m+1}V_{2m+2}} \\ & + \frac{1}{6p} + \frac{p-3}{12} - \frac{(p-1)(p-5)}{24p}, \quad (2 \nmid p), \end{aligned} \tag{6}$$

$$\begin{aligned} \sum_{n=1}^{2m} S(U_n, U_{n+1}) = & \frac{m}{6\alpha} + \frac{(p-3)m}{12} - \frac{U_{2m}}{24U_{2m+1}} - \frac{1}{24\alpha^{2m+1}U_{2m+1}} \\ & + \frac{1}{24\alpha} + \frac{1}{6}\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n}U_{2n}} + 0\left(\frac{1}{\alpha^{4m}}\right), \end{aligned} \tag{7}$$

$$\begin{aligned} \sum_{n=1}^{2m+1} S(U_n, U_{n+1}) = & \frac{m}{6\alpha} + \frac{(p-3)(2m+1)}{24} - \frac{U_{2m+1}}{24U_{2m+2}} - \frac{1}{24\alpha^{2m+2}U_{2m+2}} \\ & + \frac{1}{24\alpha} - \frac{U_{2m}}{12pU_{2m+2}} + \frac{(p-1)(p-2)}{24p} + \frac{1}{6}\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n}U_{2n}} + 0\left(\frac{1}{\alpha^{4m}}\right), \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{n=1}^{2m} S(V_n, V_{n+1}) = & \frac{(p-1)(p-5)}{48p} + \frac{m}{6\alpha} + \frac{(p-3)m}{12} \\ & - \frac{\Delta-1}{12\sqrt{\Delta}}\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n}V_{2n}} + \frac{\Delta+1}{12\sqrt{\Delta}}\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n+1}V_{2n+1}} + 0\left(\frac{1}{\alpha^{4m}}\right), \quad (2 \nmid p), \end{aligned} \tag{9}$$

$$\sum_{n=1}^{2m+1} S(V_n, V_{n+1}) = -\frac{(p-1)(p-5)}{48p} + \frac{(p-3)(m+1)}{12} + \frac{1}{6p} + \frac{m}{6\alpha} + \frac{U_{2m}}{12pV_{2m+1}} + \frac{V_{2m+1}}{12V_{2m+2}}$$
$$+ \frac{1}{12V_{2m+1}V_{2m+2}} - \frac{\Delta-1}{12\sqrt{\Delta}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n}V_{2n}}$$
$$+ \frac{\Delta+1}{12\sqrt{\Delta}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n+1}V_{2n+1}} + 0\left(\frac{1}{\alpha^{4m}}\right), \quad (2 \nmid p). \tag{10}$$

Proof: We can show that $(U_m, U_{m+1}) = 1$ $(m \geq 1)$ by induction. From the reciprocity formula of Dedekind sums (see [1]), we obtain

$$S(U_m, U_{m+1}) + S(U_{m+1}, U_m) = \frac{U_m^2 + U_{m+1}^2 + 1}{12U_mU_{m+1}} - \frac{1}{4}.$$

By using (1) and (2), we get $S(U_{m+1}, U_m) = S(U_{m-1}, U_m)$. Thus,

$$S(U_m, U_{m+1}) + S(U_{m-1}, U_m) = \frac{1}{12}\left(\frac{U_{m-1}}{U_m} + \frac{U_m}{U_{m+1}} + \frac{1}{U_mU_{m+1}}\right) - \frac{1}{4} + \frac{p}{12}$$
$$= \frac{1}{12U_mU_{m+1}} - \frac{U_{m-1}}{12pU_{m+1}} - \frac{U_{m-2}}{12pU_m} - \frac{1}{4} + \frac{p}{12} + \frac{1}{6p} \tag{11}$$

so that

$$S(U_m, U_{m+1}) + \frac{U_{m-1}}{12pU_{m+1}} = \frac{1}{12U_mU_{m+1}} - \left[S(U_{m-1}, U_m) + \frac{U_{m-2}}{12pU_m}\right] - \frac{1}{4} + \frac{p}{12} + \frac{1}{6p}$$
$$= \frac{1}{12U_mU_{m+1}} - \frac{1}{12U_{m-1}U_m} + \left[S(U_{m-2}, U_{m-1}) + \frac{U_{m-3}}{12pU_{m-1}}\right].$$

Hence,

$$S(U_{2m}, U_{2m+1}) + \frac{U_{2m-1}}{12pU_{2m+1}} = \frac{1}{12U_{2m}U_{2m+1}} - \frac{1}{12U_{2m-1}U_{2m}} + \frac{1}{12U_{2m-2}U_{2m-1}}$$
$$- \cdots + \frac{1}{12U_2U_3} - \left[S(U_1, U_2) + \frac{U_0}{12pU_2}\right] - \frac{1}{4} + \frac{p}{12} + \frac{1}{6p},$$

and

$$S(U_{2m+1}, U_{2m+2}) + \frac{U_{2m}}{12pU_{2m+2}} = \frac{1}{12U_{2m+1}U_{2m+2}} - \frac{1}{12U_{2m}U_{2m+1}} + \frac{1}{12U_{2m-1}U_{2m}}$$
$$- \cdots + \left[S(U_1, U_2) + \frac{U_0}{12pU_2}\right].$$

From the definition of U_n, we can prove that

$$\frac{1}{\alpha^m U_m} + \frac{1}{\alpha^{m+1} U_{m+1}} = \frac{1}{U_m U_{m+1}}. \tag{12}$$

On the other hand, $S(U_1, U_2) = S(1, p) = \frac{(p-1)(p-2)}{12p}$ and $U_0 = 0$. Therefore, we have (3) and (4).

Using the same method for getting (3-4), and the formula $S(V_0, V_1) = S(2, p) = \frac{(p-1)(p-5)}{24p}$ $(2 \nmid p)$ in the meantime, we can show that (5) and (6) hold.

Summing both sides of (11), we obtain

$$\sum_{n=1}^{m} [S(U_n, U_{n+1}) + S(U_{n-1}, U_n)] = 2\sum_{n=1}^{m} S(U_n, U_{n+1}) - S(U_m, U_{m+1})$$

$$= \frac{1}{12}\sum_{n=1}^{m}\left(\frac{U_n}{U_{n+1}} + \frac{U_{n-1}}{U_n} + \frac{1}{U_n U_{n+1}}\right) - \frac{m}{4} + \frac{pm}{12}$$

$$= \frac{1}{6}\sum_{n=1}^{m}\frac{U_n}{U_{n+1}} + \frac{1}{12}\sum_{n=1}^{m}\frac{1}{U_n U_{n+1}} - \frac{U_m}{12U_{m+1}} - \frac{m}{4} + \frac{pm}{12}.$$

It follows from the definition of U_n that $\sum_{n=1}^{m}\frac{U_n}{U_{n+1}} = \frac{1}{\alpha}\sum_{n=1}^{m}\frac{U_{n+1}-(-1/\alpha)^n}{U_{n+1}}$. Then

$$\sum_{n=1}^{m} S(U_n, U_{n+1}) = \frac{S(U_m, U_{m+1})}{2} + \frac{1}{12}\sum_{n=1}^{m}\frac{U_n}{U_{n+1}} + \frac{1}{24}\sum_{n=1}^{m}\frac{1}{U_n U_{n+1}} - \frac{U_m}{24U_{m+1}} + \frac{(p-3)m}{24}$$

$$= \frac{S(U_m, U_{m+1})}{2} + \frac{m}{12\alpha} + \frac{(p-3)m}{24} + \frac{1}{12}\sum_{n=1}^{m}\frac{(-1)^{n+1}}{\alpha^{n+1}U_{n+1}} - \frac{U_m}{24U_{m+1}} + \frac{1}{24}\sum_{n=1}^{m}\frac{1}{U_n U_{n+1}}.$$

By (12), we have

$$\sum_{n=1}^{m} S(U_n, U_{n+1}) = \frac{S(U_m, U_{m+1})}{2} + \frac{m}{12\alpha} + \frac{(p-3)m}{24} - \frac{U_m}{24U_{m+1}} + \frac{1}{12}\sum_{n=1}^{m}\frac{(-1)^{n+1}}{\alpha^{n+1}U_{n+1}}$$

$$+ \frac{1}{24}\sum_{n=1}^{m}\left(\frac{1}{\alpha^n U_n} + \frac{1}{\alpha^{n+1}U_{n+1}}\right).$$

Due to $\sum_{n=1}^{m}\frac{1}{\alpha^n U_n} = \sum_{n=1}^{\infty}\frac{1}{\alpha^n U_n} + 0\left(\frac{1}{\alpha^{2m}}\right)$, the equalities (7-8) hold.

Similarly, we can prove that (9) and (10) hold. □

Theorem 2: Suppose that $q = 1$ and m is a positive integer. Then

$$S(U_m, U_{m+1}) = \frac{U_m}{6U_{m+1}} + \frac{m(p-3)}{12}, \tag{13}$$

$$S(V_m, V_{m+1}) = \frac{p-6}{24} + \frac{V_m}{12V_{m+1}} + \frac{(p-3)m}{12} + \frac{1}{12\sqrt{\Delta}}\left(\frac{1}{2} - \frac{1}{\alpha^{m+1}V_{m+1}}\right) \quad (2 \not| \, p), \qquad (14)$$

$$\sum_{n=1}^{m} S(U_n, U_{n+1}) = \frac{(p-3)m(m+1)}{24} + \frac{m}{6\alpha} - \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{\alpha^{n+1}U_{n+1}} + 0\left(\frac{1}{\alpha^{2m}}\right), \qquad (15)$$

$$\sum_{n=1}^{m} S(V_n, V_{n+1}) = \frac{[2p-9+(p-3)m]m}{24} + \frac{m}{12\alpha} + \frac{m}{24\sqrt{\Delta}}$$

$$+ \frac{\Delta - 1}{12\sqrt{\Delta}}\sum_{n=1}^{\infty}\frac{1}{\alpha^{n+1}V_{n+1}} + 0\left(\frac{1}{\alpha^{2m}}\right). \qquad (16)$$

Proof: One can verify that $(U_m, U_{m+1}) = 1 (m \geq 1)$ by induction. So,

$$S(U_m, U_{m+1}) + S(U_{m+1}, U_m) = \frac{U_m^2 + U_{m+1}^2 + 1}{12U_mU_{m+1}} - \frac{1}{4}.$$

It follows from (1-2) and $((-x)) = -((x))$ that

$$S(U_{m+1}, U_m) = -S(U_{m-1}, U_m).$$

Thus, the following identity holds:

$$S(U_m, U_{m+1}) - S(U_{m-1}, U_m) = \frac{1}{12}\left(\frac{U_m}{U_{m+1}} - \frac{U_{m-1}}{U_m} + \frac{1}{U_mU_{m+1}}\right) + \frac{p}{12} - \frac{1}{4}. \qquad (17)$$

Summing both sides of (17) and noticing that $S(U_0, U_1) = 0$, we get

$$S(U_m, U_{m+1}) = \frac{U_m}{12U_{m+1}} + \frac{(p-3)m}{12} + \frac{1}{12}\sum_{n=1}^{m}\frac{1}{U_nU_{n+1}}.$$

From [2] or [3], we know that $\sum_{n=1}^{m}\frac{1}{U_nU_{n+1}} = \frac{\alpha^mU_{m+1}-1}{\alpha^{m+1}U_{m+1}}$. Hence,

$$S(U_m, U_{m+1}) = \frac{\alpha^{m+1}U_m + \alpha^mU_{m+1} - 1}{12\alpha^{m+1}U_{m+1}} + \frac{(p-3)m}{12}.$$

By the definition of U_n, we have $\frac{\alpha^{m+1}U_m+\alpha^mU_{m+1}-1}{\alpha^{m+1}U_m} = \frac{2U_m}{U_{m+1}}$. Hence, equality (13) holds.

By (13), we have

$$\sum_{n=1}^{m} S(U_n, U_{n+1}) = \frac{(p-3)m(m+1)}{24} + \frac{1}{6}\sum_{n=1}^{m}\frac{U_n}{U_{n+1}}.$$

From the definition of U_n, we can deduce that

$$\frac{\alpha U_n}{U_{n+1}} = \frac{U_{n+1} - \beta^n}{U_{n+1}} = 1 - \frac{1}{\alpha^nU_{n+1}}.$$

Therefore, equality (15) holds.

Similarly, we can show that (14) and (16) hold. □

We note that (7) and (8) generalize Zhang's results (see [7]):

$$\sum_{n=1}^{m} S(F_n, F_{n+1}) = -\frac{(\sqrt{5}-1)^2 m}{48} + C(m) + 0\left(\frac{2^{2m}}{(\sqrt{5}+1)^{2m}}\right),$$

where

$$C(m) = \begin{cases} \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+1}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{(\sqrt{5}+1)^{n+1}F_{n+1}}, & \text{if } m \text{ is an even number;} \\ \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}F_{2n+2}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{(\sqrt{5}+1)^{n+1}F_{n+1}}, & \text{if } m \text{ is an odd number,} \end{cases}$$

(in [7],

$$C(m) = \begin{cases} \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+1}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{2^{n+1}}{(\sqrt{5}+1)^{n+1}F_{n}}, & \text{if } m \text{ is an even number;} \\ \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}F_{2n+2}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{2^{n+1}}{(\sqrt{5}+1)^{n+1}F_{n}}, & \text{if } m \text{ is an odd number,} \end{cases}$$

in which there exist printing errors, 2^{n+1} and F_n should be $(-2)^{n+1}$ and F_{n+1}, respectively).

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