ON PERIODIC ∞-GENERALIZED FIBONACCI SEQUENCES

B. Bernoussi

Univiersité Abdelmalek Essaadi, ENSA de Tanger, B. P. 416, Tanger, Morocco
e-mail: Benaissa@fstt.ac.ma

W. Motta

Faculdade de Matemática, UFU, Campus Santa Mônica, 38408-100 Uberlândia, MG, Brazil
e-mail: wmotta@ufu.br

M. Rachidi

Département de Mathématiques, Faculté des Sciences, Université Mohammed V, B. P. 1014, Rabat, Morocco
e-mail: rachidi@fsr.ac.ma

O. Saeki

Faculty of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8581, Japan
e-mail: saeki@math.kyushu-u.ac.jp

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1. INTRODUCTION

The notion of an ∞-generalized Fibonacci sequence has been introduced in [6], and studied in [1], [7], [9]. This class of sequences defined by linear recurrences of infinite order is an extension of the class of ordinary (weighted) r-generalized Fibonacci sequences (r-GFS, for short) with r finite defined by linear recurrences of r^{th} order (for example, see [2], [3], [4], [5], [8] etc.) More precisely, let $\{a_i\}_{i=0}^{\infty}$ and $\{\alpha_{-i}\}_{i=0}^{\infty}$ be two sequences of real or complex numbers, where $a_i \neq 0$ for some i. The former is called the *coefficient sequence* and the latter the *initial sequence*. The associated ∞-*generalized Fibonacci sequence* (∞-GFS, for short) $\{V_n\}_{n\in\boldsymbol{Z}}$ is defined as follows:

$$V_n = \alpha_n \qquad \text{if } n \leq 0, \tag{1.1}$$

$$V_n = \sum_{i=0}^{\infty} a_i V_{n-i-1} \qquad \text{if } n \geq 1. \tag{1.2}$$

As is easily observed, the general terms V_n may not necessarily exist. In [1], necessary and sufficient conditions for the existence of the general terms have been studied.

In this paper, we consider the case where the coefficient sequence is periodic. In §2, we give several necessary and sufficient conditions for the existence of the general terms V_n $(n \geq 1)$ of such a sequence. We will also see that, in our case, the sequence $\{V_n\}_{n\geq 1}$ can be considered as an r-GFS with r being the period of the coefficient sequence, where the new associated finite sequence is obtained from the original periodic infinite sequence by a slight modification (see Theorem 2.4). In §3, we consider a power series associated with the infinite coefficient sequence, which plays a role similar to that of the characteristic polynomial for an r-GFS with r finite. We will see that the inverses of the zeros of such a power series are roots of the characteristic polynomial associated with the modified finite coefficient sequence. Finally in §4, we will confine ourselves to the case where $a_i \geq 0$ and obtain some results about the asymptotic behavior of such sequences.

2. EXISTENCE OF GENERAL TERMS

Let $\{a_i\}_{i=0}^{\infty}$ and $\{\alpha_{-i}\}_{i=0}^{\infty}$ be as in §1 and $\{V_n\}_{n\in\boldsymbol{Z}}$ the associated ∞-GFS defined by (1.1) and (1.2). Throughout the paper, we assume that there exists a positive integer r such that

$$a_{i+r} = a_i \quad \text{for all } i \geq 0. \tag{2.1}$$

In other words, we assume that the coefficient sequence $\{a_i\}$ is periodic of period $r > 0$. In this case, we call the sequence $\{V_n\}_{n\in\boldsymbol{Z}}$ a *periodic ∞-generalized Fibonacci sequence*. In this section, we give some necessary and sufficient conditions for the existence of the general terms of a periodic ∞-GFS. First we show the following.

Proposition 2.1: *If the series*

$$S_j = \sum_{k=0}^{\infty} \alpha_{-kr-j} \tag{2.2}$$

converges for all j with $0 \leq j \leq r-1$, then the general term V_n exists for all $n \geq 1$.

Proof: By [1], we have only to show that for all $n \geq 1$, the series

$$\sum_{i=0}^{\infty} a_{i+n-1}\alpha_{-i} \tag{2.3}$$

converges. First note that

$$\lim_{i\to\infty} \alpha_{-i} = 0 \tag{2.4}$$

by our assumption. For $m \geq 0$ and j with $0 \leq j \leq r-1$, we have

$$\begin{aligned}\sum_{i=0}^{mr+j} a_{i+n-1}\alpha_{-i} &= \sum_{k=0}^{m-1}\sum_{l=0}^{r-1} a_{kr+l+n-1}\alpha_{-(kr+l)} + \sum_{i=mr}^{mr+j} a_{i+n-1}\alpha_{-i} \\ &= \sum_{l=0}^{r-1} a_{l+n-1}\left(\sum_{k=0}^{m-1}\alpha_{-kr-l}\right) + \sum_{i=0}^{j} a_{i+n-1}\alpha_{-(i+mr)},\end{aligned}$$

where the second equality follows from (2.1). Then by our assumption together with (2.4), we see that

$$\lim_{m\to\infty} \sum_{i=0}^{mr+j} a_{i+n-1}\alpha_{-i} = \sum_{l=0}^{r-1} a_{l+n-1}S_l$$

for every j with $0 \leq j \leq r-1$, where the limiting value does not depend on j. Hence the series (2.3) converges. This completes the proof. $\square$

Let us consider the converse of Proposition 2.1. Consider the polynomial $T(x) = \sum_{j=0}^{r-1} a_j x^{r-1-j}$. Then we have the following.

Proposition 2.2: *If $T(x)$ does not have any root $\xi \in \boldsymbol{C}$ with $\xi^r = 1$, then the general term V_n exists for all $n \geq 1$ if and only if the series (2.2) converges for all j with $0 \leq j \leq r-1$.*

Proof: Consider the matrix

$$M = \begin{pmatrix} a_{r-1} & a_{r-2} & \cdots & a_1 & a_0 \\ a_0 & \ddots & \ddots & & a_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{r-3} & & \ddots & \ddots & a_{r-2} \\ a_{r-2} & a_{r-3} & \cdots & a_0 & a_{r-1} \end{pmatrix}.$$

Then it is easy to see that $M = \sum_{j=0}^{r-1} a_j B^{r-1-j}$, where B is the $r \times r$ matrix given by

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Since B is diagonalizable over the complex numbers and its eigenvalues are the r^{th} roots of unity, we see that M is nonsingular if and only if $T(\xi) \neq 0$ for all r^{th} roots ξ of unity, which is satisfied by our hypothesis. Hence M is nonsingular and has its inverse. Thus for each j with $0 \leq j \leq r-1$, there exists a set of r complex numbers $v_0^{(j)}, \ldots, v_{r-1}^{(j)}$ such that for all k with $0 \leq k \leq r-1$, we have $\sum_{l=0}^{r-1} v_l^{(j)} a_{l+k} = \delta_{kj}$, where

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Now let us assume that the general terms V_n exist. Then by [1], the series (2.3) converges for all n. Hence, putting

$$\tilde{\delta}_{kj} = \begin{cases} 1 & \text{if } k \equiv j \bmod r, \\ 0 & \text{if } k \not\equiv j \bmod r, \end{cases}$$

we have, for each j with $0 \leq j \leq r-1$,

$$\sum_{l=0}^{r-1} v_l^{(j)} \sum_{i=0}^{\infty} a_{i+l} \alpha_{-i} = \sum_{i=0}^{\infty} \left(\sum_{l=0}^{r-1} v_l^{(j)} a_{l+i} \right) \alpha_{-i} = \sum_{i=0}^{\infty} \tilde{\delta}_{ij} \alpha_{-i},$$

which shows that the sequence (2.2) converges. The converse follows from Proposition 2.1. This completes the proof. □

We also have the following.

Proposition 2.3: *The general term V_n exists for all $n \geq 1$ if and only if the first r terms $V_1, \ldots, V_r$ exist.*

Proof: Suppose that the first r terms $V_1, \dots, V_r$ exist. Let us show that the terms $V_1, \dots, V_n$ exist for all n by induction on n. When $n = r$, this is obvious. Suppose $n \geq r$ and that $V_1, \dots, V_n$ exist. Then by (1.2), we have

$$V_{n+1} = \sum_{i=0}^{r-1} a_i V_{n-i} + \sum_{i=r}^{\infty} a_i V_{n-i} \tag{2.5}$$

$$= \sum_{i=0}^{r-1} a_i V_{n-i} + \sum_{i=0}^{\infty} a_{i+r} V_{n-i-r} \tag{2.6}$$

$$= \sum_{i=0}^{r-1} a_i V_{n-i} + \sum_{i=0}^{\infty} a_i V_{n-i-r} \tag{2.7}$$

$$= \sum_{i=0}^{r-1} a_i V_{n-i} + V_{n-r+1} \tag{2.8}$$

$$= \sum_{i=0}^{r-2} a_i V_{n-i} + (a_{r-1} + 1) V_{n-r+1}, \tag{2.9}$$

where (2.7) follows from (2.1) and (2.8) follows from (1.2) and our induction hypothesis. Thus the term V_{n+1} exists. This completes the proof. □

Note that (2.9) shows the following.

Theorem 2.4: *If the first r terms $V_1, \dots, V_r$ exist, then the sequence $\{V_n\}_{n\geq 1}$ is a (weighted) r-generalized Fibonacci sequence with respect to the coefficient sequence*

$$\{a_0, a_1, \dots, a_{r-2}, a_{r-1} + 1\}$$

and the initial sequence $\{V_1, \dots, V_r\}$ in the sense of [3].

In particular, we have a Binet type formula (see [3, Theorem 1]) and consequently we can obtain some information about the asymptotic behavior of such sequences using results of [3] (and without using results of [6] about ∞-GFS's).

Remark 2.5: So far, we have not assumed that r is the minimum positive integer satisfying (2.1). Thus Theorem 2.4 holds even if we replace r with kr, where k is an arbitrary positive integer.

3. CHARACTERISTIC ROOTS

In this section, for a periodic ∞-GFS, we consider an analogue of the characteristic polynomial for an r-GFS with r finite. Consider the series $Q(z)$ associated with the coefficient sequence $\{a_i\}_{i=0}^{\infty}$ defined by $Q(z) = 1 - \sum_{i=0}^{\infty} a_i z^{i+1}$ (see [1, §5]). In the following, we suppose that $a_{i_0} \neq 0$ for some $i_0 \geq 0$.

Lemma 3.1: *The radius of convergence of the series $Q(z)$ is equal to 1. In fact, for a complex number z, the series $Q(z)$ converges if and only if $|z| < 1$.*

Proof: Suppose that the series $Q(z)$ converges for a complex number z. Then we have $\lim_{i\to\infty} a_i z^{i+1} = 0$. Since $a_i \neq 0$ for all $i \equiv i_0 \bmod r$, this implies that $\lim_{i\to\infty} z^{i+1} = 0$. Hence we have $|z| < 1$. The converse is obvious, since the sequence $\{a_i\}_{i=0}^{\infty}$ is bounded. This completes the proof. $\square$

For $m \geq 0$, set $Q_m(z) = 1 - \sum_{i=0}^{m} a_i z^{i+1}$. Suppose $|z| < 1$. Then we have $Q(z) = \lim_{m\to\infty} Q_m(z)$ and in particular

$$Q(z) = \lim_{k\to\infty} Q_{kr-1}(z). \tag{3.1}$$

By (2.1), we see easily that

$$\begin{aligned}
Q_{kr-1}(z) &= 1 - \sum_{i=0}^{kr-1} a_i z^{i+1} \\
&= 1 - \sum_{l=0}^{k-1} \left(\sum_{j=0}^{r-1} a_j z^{lr+j+1} \right) \\
&= 1 - \sum_{j=0}^{r-1} a_j z^{j+1} \sum_{l=0}^{k-1} z^{lr} \\
&= 1 - \left(\sum_{j=0}^{r-1} a_j z^{j+1} \right) \frac{1 - z^{kr}}{1 - z^r}.
\end{aligned}$$

Thus we have, by (3.1),

$$Q(z) = \frac{P_1(z)}{1 - z^r}, \tag{3.2}$$

since $|z| < 1$, where

$$P_1(z) = 1 - \sum_{j=0}^{r-2} a_j z^{j+1} - (a_{r-1} + 1) z^r.$$

Let P be the characteristic polynomial for an r-GFS with respect to the coefficient sequence $\{a_0, \ldots, a_{r-2}, a_{r-1} + 1\}$ given by

$$P(x) = x^r - a_0 x^{r-1} - a_1 x^{r-2} - \cdots - a_{r-2} x - (a_{r-1} + 1). \tag{3.3}$$

Note that $P(x) = x^r P_1(x^{-1})$ and hence by (3.2) we have

$$Q(x^{-1}) = \frac{P(x)}{x^r - 1}. \tag{3.4}$$

Thus, for a complex number λ with $|\lambda| > 1$, $Q(\lambda^{-1}) = 0$ if and only if $P(\lambda) = 0$ by Lemma 3.1 and (3.4). Thus we have the following.

Proposition 3.2: *For a nonzero complex number λ, if λ^{-1} is a zero of Q, then $|\lambda| > 1$ and λ is a root of the characteristic polynomial P. Conversely, if λ is a root of P with $|\lambda| > 1$, then λ^{-1} is a zero of Q.*

Remark 3.3: If λ is as in Proposition 3.2, then the sequence $\{W_n\}_{n\in\boldsymbol{Z}}$ defined by $W_n = \lambda^n$ is a periodic ∞-GFS associated with the coefficient sequence $\{a_i\}_{i=0}^{\infty}$.

Proposition 3.2 shows that the inverse of every zero of Q is a root of the characteristic polynomial P. The converse does not hold in general as is seen in the following example.

Example 3.4: Consider the coefficient sequence $\{a_i\}_{i=0}^{\infty}$ defined by

$$a_i = \begin{cases} 4/3 & \text{if } i \equiv 0 \bmod 2, \\ 1/3 & \text{if } i \equiv 1 \bmod 2, \end{cases}.$$

where $r = 2$. Then we see easily that the roots of the characteristic polynomial P defined by (3.3) are $\lambda_1 = 2$ and $\lambda_2 = -2/3$. Thus λ_1^{-1} is a zero of Q, while λ_2^{-1} is not. In particular, the sequence $\{W_n\}_{n\in\boldsymbol{Z}}$ defined by $W_n = \lambda_2^n$ is *not* an ∞-GFS associated with the coefficient sequence $\{a_i\}_{i=0}^{\infty}$, since the series in (1.2) does not converge, while it *is* a 2-GFS associated with the coefficient sequence $\{a_0, a_1 + 1\}$.

4. THE CASE OF NONNEGATIVE COEFFICIENTS

Throughout this section, we assume that all the coefficients a_i are nonnegative real numbers. By [3, Lemma 8], there exists a unique positive root q of the characteristic polynomial P defined by (3.3).

Lemma 4.1: *We always have $q > 1$.*

Proof: The function φ defined by $\varphi(x) = x^{-r}P(x)$ for $x > 0$ is obviously differentiable and increasing. Furthermore, we have $\lim_{x\to+0}\varphi(x) = -\infty$, $\lim_{x\to\infty}\varphi(x) = 1$ and $\varphi'(x) > 0$ for all $x > 0$. Thus there exists a unique positive zero of φ, which is nothing but the root q of P. Now, using our assumption together with (3.3), we see easily that $\varphi(1) < 0$, which implies that $q > 1$. This completes the proof. $\square$

The above lemma together with Proposition 3.2 shows that q^{-1} is a zero of the series Q. Thus the conditions of [1, §5] are satisfied and the results obtained there can be applied. In the following, let us assume that the general term V_n exists for all $n \geq 1$. Then, we have the following.

Proposition 4.2: *The following three conditions are equivalent.*

(1) *$\lim_{n\to\infty} V_n/q^n$ exists.*

(2) *The greatest common divisor GCD_∞ of the integers $\{i + 1 : a_i \neq 0\}$ is equal to 1.*

(3) *The greatest common divisor GCD_r of the integers $\{i + 1 : a_i \neq 0, 0 \leq i \leq r - 2\} \cup \{r\}$ is equal to 1.*

Proof: The equivalence of (1) and (3) follows from the results of [3, §4], since, by Theorem 2.4, $\{V_n\}_{n=1}^{\infty}$ is an r-GFS associated with the coefficient sequence $\{a_0, \dots, a_{r-2}, a_{r-1}+1\}$ (see also [10, Theorem 12.2]. By (2.1), we see that GCD_∞ coincides with the greatest common divisor of the integers $\{i+1 : a_i \neq 0, 0 \leq i \leq r-1\} \cup \{r\}$ and hence with GCD_r. From this the equivalence of (2) and (3) follows. This completes the proof. $\square$

Note that a formula for the limiting value of Proposition 4.2 (1) is obtained in [3] (see also [6]).

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