ON PERIODIC SOLUTIONS OF A CERTAIN DIFFERENCE EQUATION

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ABSTRACT

We give a method for finding periodic solutions of the equation $y_{n+1} = \lambda T_n\left(\dfrac{y_n}{\lambda}\right)$, where T_n, $n \in \mathbb{N}$, is a Chebyshev polynomial of the first kind and degree n. Some consideration of Lucas and Mersenne numbers is also given.

1. INTRODUCTION AND PRELIMINARIES

A method for finding periodic solutions of the logistic difference equation

$$x_{n+1} = \lambda x_n(1 - x_n), \quad x_n \in \mathbb{R},\ \lambda \in \mathbb{R},\ n \in \mathbb{Z} \tag{1}$$

was given in [1]. More precisely, for a given parameter λ, an initial condition generating the periodic solution is determined.

If a linear substitution $x_n = 1/2 - y_n/\lambda$ is introduced, one obtains a canonical form

$$y_{n+1} = y_n^2 - b \tag{2}$$

of the equation (1), where $b = \lambda^2/4 - \lambda/2$. A recurrence relation $L_{n+1} = L_n^2 - 2$ (with an initial value $L_1 = 4$), which is a canonical form of the logistic equation for $b = 2$, defines the Lucas numbers. We can rewrite it as

$$y_{n+1} = y_n^2 - 2 = 2\left(2\left(\frac{y_n}{2}\right)^2 - 1\right) = 2T_2\left(\frac{y_n}{2}\right), \tag{3}$$

where $T_2(x) = 2x^2 - 1$ is a Chebyshev polynomial. So, we were motivated by this fact to consider (3) as a special case of a more general equation

$$y_{n+1} = \lambda T_r\left(\frac{y_n}{\lambda}\right), \tag{4}$$

where r is a prime number, $\lambda \in \mathbb{R}$.

2. PERIODIC SOLUTION OF THE EQUATION (4)

A periodic solution of a difference equation is the one satisfying condition $y_{n+p} = y_n$, where $p \in \mathbb{N}$, so that $y_{n+q} \neq y_n$ whenever $1 \leq q < p$. We call p *a period.* Any other solution

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is called a non-periodic solution. *A trivial solution* is $y_n = 0$. In order to find all periodic solutions of the equation (4), we will first find periodic solutions of the equation

$$y_{n+1} = y_n^r, \quad r \text{ is a prime number.} \tag{5}$$

Lemma 1: *A general solution of the equation* (5) *is*

$$y_n = a^{r^n},$$

where $y_0 = a \in \mathbb{C}$ is an arbitrary initial value.

Proof: It immediately follows from

$$y_n = y_{n-1}^r = (y_{n-2}^r)^r = y_{n-2}^{r^2} = \cdots = y_0^{r^n}. \ \square$$

Theorem 1: *All periodic nontrivial solutions of the equation* (5) *for a period p are given by*

$$\prod_{d|r^p-1} \Phi_d(y) = 0,$$

where $d \nmid r^q - 1$ for $q \mid p$, $q < p$ and $\Phi_d(y)$ are cyclotomic polynomials

$$\Phi_d(y) = \prod_{l|d} (y^l - 1)^{\mu(d/l)}, \quad y \in \mathbb{C}, \quad d \in \mathbb{N},$$

and μ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1\,, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 \cdots p_k, \text{ where } p_i \text{ are distinct primes} \\ 0\,, & \text{if } p^2 \mid n \text{ for some prime } p. \end{cases}$$

Proof: Periodic solutions must satisfy the relation $y_{n+p} = y_n$. By Lemma 1 we have

$$a^{r^{n+p}} = a^{r^n} \ \Rightarrow\ \left(a^{r^n}\right)^{r^p} - a^{r^n} = 0 \ \Rightarrow\ y_n\left(y_n^{r^p-1} - 1\right) = 0.$$

If $q \mid p,\ q < p$ there follows $r^q - 1 \mid r^p - 1$, which in turn implies

$$y_n^{r^q-1} - 1 \mid y_n^{r^p-1} - 1. \tag{6}$$

Because of (see Lidl, Niederreiter [2])

$$z^n - 1 = \prod_{d|n} \Phi_d(z)$$

we have

$$y_n\left(y_n^{r^p-1} - 1\right) = y_n \prod_{d|r^p-1} \Phi_d(y_n) = 0. \tag{7}$$

So all periodic nontrivial solutions for the period p are given by $\Phi_d(y) = 0$ where $d \mid r^p - 1$, and $d \nmid r^q - 1$ for $q \mid p$, $q < p$. We had to exclude a product of cyclotomic polynomials Φ_d where $d \mid r^q - 1$, $q \mid p$, $q < p$, because, considering (6), the equation

$$y_n^{r^q-1} - 1 = \prod_{\substack{d \mid r^q-1 \\ q \mid p, q<p}} \Phi_d(y_n) = 0$$

would give nontrivial solutions of the equation $y_n^{r^q} = y_n$ obtained as a result of the relation $y_{n+q} = y_n$. That means we have required periodic solutions for the periods $q \mid p$, $q < p$, which is untrue. □

Example 1: Let us consider the period $p = 6$. In that case $2^p - 1 = 63$. Divisors of 63 are 1, 3, 7, 9, 21, 63. So we have

$$y_n\left(y_n^{63} - 1\right) = y_n \prod_{d \mid 63} \Phi_d(y_n)$$

$$= \Phi_0(y_n)\Phi_1(y_n)\Phi_3(y_n)\Phi_7(y_n)\Phi_9(y_n)\Phi_{21}(y_n)\Phi_{63}(y_n) = 0,$$

where $\Phi_0(y) = y$ and $\Phi_1(y) = y - 1$. However, divisors q of 6, $q < 6$ are 1, 2, 3. There follows we must exclude $2^1 - 1 = 1$, $2^2 - 1 = 3$, $2^3 - 1 = 7$, which means we omit a product of those polynomials roots of which are periodic solutions for periods 1, 2, 3, that is the product of cyclotomic polynomials $\Phi_1(y), \Phi_3(y), \Phi_7(y)$. All periodic nontrivial solutions for the period $p = 6$ are obtained as roots of the cyclotomic polynomials $\Phi_9(y) = y^6 + y^3 + 1, \Phi_{21}(y) = y^{12} - y^{11} + y^9 - y^8 + y^6 - y^4 + y^3 - y + 1, \Phi_{63}(y) = y^{36} - y^{33} + y^{27} - y^{24} + y^{18} - y^{12} + y^9 - y^3 + 1$.

For some values of p and r, periodic solutions are obtained by means of the equations

$$\begin{aligned}
r = 3,\ p = 1: &\quad \Phi_1 = 0,\ \Phi_2 = 0; \\
p = 2: &\quad \Phi_4 = 0,\ \Phi_8 = 0; \\
p = 3: &\quad \Phi_{13} = 0,\ \Phi_{26} = 0. \\
r = 5,\ p = 1: &\quad \Phi_1 = 0,\ \Phi_2 = 0,\ \Phi_4 = 0; \\
p = 2: &\quad \Phi_3 = 0,\ \Phi_6 = 0,\ \Phi_8 = 0,\ \Phi_{12} = 0,\ \Phi_{24} = 0. \quad \square
\end{aligned}$$

Note that on the basis of

$$\deg \Phi_n = \sum_{d \mid n} d\mu\left(\frac{n}{d}\right) = \varphi(n),$$

one can determine the degree of a cyclotomic polynomial, where φ is Euler's function.

Now we are going to find periodic solutions of the difference equation (4).

Lemma 2: *A general solution of the equation* (4) *is given by*

$$y_n = \lambda T_{r^n}\left(\frac{y_0}{\lambda}\right),$$

where $y_0 = a \in \mathbb{C}$ is an arbitrary initial value.

Proof: Chebyshev polynomials $T_n(x)$ are defined by

$$T_n(x) = \cos(n \arccos x),$$

whence there follows

$$\begin{aligned} T_{mn}(x) &= \cos(mn \arccos x) = \cos(m(n \arccos x)) \\ &= \cos(m \arccos(\cos(n \arccos x))) \\ &= \cos(m \arccos(T_n(x))) = T_m(T_n(x)). \end{aligned}$$

By using this property we easily find

$$y_n = \lambda T_r\left(\frac{y_{n-1}}{\lambda}\right) = \lambda T_r\left(T_r\left(\frac{y_{n-2}}{\lambda}\right)\right) = \lambda T_{r^2}\left(\frac{y_{n-2}}{\lambda}\right) = \cdots = \lambda T_{r^n}\left(\frac{y_0}{\lambda}\right). \quad \square$$

Theorem 2: *All periodic nontrivial solutions for a period p can be found from an equation expressed in the form of*

$$2^k \prod_{\substack{d \mid r^p-1 \\ d \nmid r^q-1}} \Phi_d(x) = Q(x),$$

where the polynomial $Q(x)$ and $k \in \mathbb{N}$ are to be determined.

Proof: For a period p, periodic solutions are obtained by means of the relation $y_{n+p} = y_n$. By Lemma 2 we have

$$y_{n+p} = \lambda T_{r^{n+p}}\left(\frac{y_0}{\lambda}\right) = y_n = \lambda T_{r^n}\left(\frac{y_0}{\lambda}\right) \;\Rightarrow\; \lambda T_{r^n r^p}\left(\frac{y_0}{\lambda}\right) = \lambda T_{r^n}\left(\frac{y_0}{\lambda}\right).$$

By making use of the above property $T_{mn}(x) = T_m(T_n(x))$, we find

$$\lambda T_{r^p}\left(\frac{y_n}{\lambda}\right) = y_n \;\Rightarrow\; T_{r^n}\left(\frac{y_n}{\lambda}\right) = \frac{y_n}{\lambda}.$$

Denoting $x_n = \dfrac{y_n}{\lambda}$ we come to an equation

$$T_{r^p}(x_n) = x_n. \tag{8}$$

As we know that the coefficient at x^n in Chebyshev polynomials $T_n(x)$ is 2^{n-1}, the equation (8) (when we drop the subscript n for the sake of simplicity) can be rewritten in the form of

$$2^{r^p-1}x^{r^p} - 2^{r^p-1}x = P(x) \;\Leftrightarrow\; 2^{r^p-1}x\left(x^{r^p-1} - 1\right) = P(x),$$

where $P(x)$ is a polynomial obtained after the rearrangement of (8). Considering (7) the last equation becomes

$$2^{r^p-1}\Phi_0(x) \prod_{d \mid r^p-1} \Phi_d(x) - P(x) = 0 \qquad (\Phi_0(x) = x). \tag{9}$$

All periodic solutions for the period p, including periods q such that $q \mid p,\ q < p$, can be obtained form (9). Following the line of reasoning as in the proof of Theorem 1, in order to find periodic solutions for the period p, we divide the equation (9) by the polynomials giving periodic solutions for the periods $q < p,\ q \mid p$. These polynomials have a form of the left-hand side of (9) and contain a product of the cyclotomic polynomials subscripts of which are divisors of $r^q - 1$. It means that in the product of cyclotomic polynomials on the left-hand side of (9) must be omitted those cyclotomic polynomials subscripts of which are divisors of $r^q - 1$ and after division, instead of the polynomial $P(x)$, a polynomial $Q(x)$ will appear. □

Example 2: Let $p = 4$ and $r = 2$. According to (8) we start from the equation $T_{16}(x) = x$, i.e.

$$\begin{aligned}32768x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10}\\ +84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + 1 = x.\end{aligned}$$

After a rearrangement we get

$$2^{15}x(x^{15} - 1) = P(x) \;\Leftrightarrow\; 2^{15}\Phi_0(x)\Phi_1(x)\Phi_3(x)\Phi_5(x)\Phi_{15} - P(x) = 0, \tag{10}$$

where

$$\begin{aligned}P(x) = 131072x^{14} - 212992x^{12} + 180224x^{10}\\ -84480x^8 + 21504x^6 - 2688x^4 + 128x^2 - 32767x - 1.\end{aligned}$$

The equation (10) contains all periodic solutions, but we exclude the solutions for the periods $q < 4,\ q \mid 4$, that is $q = 1$ or $q = 2$. So we have to consider now those relations giving periodic solutions for the periods 1 and 2, that is $y_{n+1} = y_n$ and $y_{n+2} = y_n$. However, the latter comprises all periodic solutions of the first one, and the equation (8) becomes

$$T_4(x) = x \;\Leftrightarrow\; 8x^4 - 8x^2 + 1 = x \;\Leftrightarrow\; 8x^4 - 8x^2 - x + 1 = 0.$$

It is necessary to divide (10) by the polynomial $8x^4 - 8x^2 - x + 1$, roots of which contain all periodic solutions for the period $q = 2$, including $q = 1$. There holds

$$x(x-1)(x^2 + x + 1) = \Phi_0(x)\Phi_1(x)\Phi_3(x),$$

and we have

$$8x^4 - 8x^2 - x + 1 = 0 \;\Leftrightarrow\; 2^3\Phi_0(x)\Phi_1(x)\Phi_3(x) - (8x^2 - 7x - 1) = 0,$$

After dividing the equation (10) by the polynomial $8x^4 - 8x^2 - x + 1$ containing the product $2^3\Phi_0(x)\Phi_1(x)\Phi_3(x)$, as a result we obtain

$$\begin{aligned}4096x^{12} - 12288x^{10} + 512x^9 + 13824x^8 - 1024x^7 - 7104x^6 + 640x^5\\ +1600x^4 - 120x^3 - 120x^2 + 1 = 0.\end{aligned}$$

Rewriting this equation we have

$$\begin{aligned}2^{12}\Phi_5(x)\Phi_{15}(x) &\equiv 4096(1 + x^3 + x^6 + x^9 + x^{12})\\ &= 12288x^{10} + 3584x^9 - 13824x^8 + 1024x^7 + 11200x^6 - 640x^5\\ &\quad -1600x^4 + 4216x^3 + 120x^2 + 4095 = Q(x). \quad \square\end{aligned}$$

Example 3: Let now $r = 3$ and $p = 2$. We start from the equation $T_9(x) = x$ and come to the equation $256x^9 - 576x^7 + 432x^5 - 120x^3 + 8x = 0$. After a rearrangement we get

$$\begin{aligned}2^8x(x^8 - 1) = P(x) &\Leftrightarrow 2^8x(x-1)(x+1)(x^2+1)(x^4+1) = P(x)\\ &\Leftrightarrow 2^8\Phi_0(x)\Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x) - P(x) = 0,\end{aligned}$$

where $P(x) = 576x^7 - 432x^5 + 120x^3 - 264x$. But we have to divide the above equation by a polynomial roots of which are periodic solutions for the period $p = 1$. In order to find that polynomial, we consider the equation $T_3(x) = x$, whence we obtain the polynomial $2^2\Phi_0(x)\Phi_1(x)\Phi_2(x) + 2x$. After dividing we obtain the equation $64x^6 - 80x^4 + 28x^2 - 2 = 0$. In other words

$$2^6\Phi_4(x)\Phi_8(x) \equiv 2^6(x^2+1)(x^4+1) = 144x^4 + 36x^2 + 66 = Q(x). \quad \square$$

3. SOME NOTES ON THE LUCAS AND MERSENNE NUMBERS

We are now coming back to the recurrence relation (3) defining the Lucas numbers. Considering that a general solution of the equation (4) is

$$y_n = 2T_{2^n}\left(\frac{y_0}{2}\right),$$

it is, at the same time, a general solution of the equation (3). By choosing $y_0 = \sqrt{6}$, seeing as $L_1 = 4$, we find a general formula for the Lucas numbers

$$L_n = 2T_{2^{n-1}}(2).$$

However, taking account of (see Suetin [4])

$$T_n(z) = \frac{1}{2}\left(\left(z + \sqrt{z^2-1}\right)^n + \left(z - \sqrt{z^2-1}\right)^n\right), \quad z \in \mathbb{C}$$

the Lucas numbers can be expressed explicitly in the following way

$$L_n = (2+\sqrt{3})^{2^{n-1}} + (2-\sqrt{3})^{2^{n-1}}.$$

Also, the well-known Lucas-Lehmer theorem (see Sierpiński [3]) concerned with a test ascertaining whether Mersenne numbers are prime or not, now becomes: *A Mersenne number M_p, p being an odd prime, is prime if and only if it is a divisor of $T_{2^{p-2}}(2)$.*

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