ON GCD-LCM DUALITY BETWEEN PASCAL'S PYRAMID AND THE MODIFIED PASCAL PYRAMID

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(Submitted April 2002-Final Revision August 2003)

ABSTRACT

Let $\Delta(P)$ be the m-dimensional Pascal's pyramid consisting of m-nomial coefficients $n!/r_1!r_2!\dots r_m!$, and $\Delta(M)$ be the modified one of the modified m-nomial coefficients $(n+m-1)!/r_1!r_2!\dots r_m!$, where $n = r_1 + r_2 + \cdots + r_m$. It will be proved that a GCD equality for a configuration C, which holds in $\Delta(P)$ corresponds to an LCM equality for C' in $\Delta(M)$, which is symmetric to C with respect to a point, and an LCM equality for C in $\Delta(P)$ to a GCD equality for C' in $\Delta(M)$. The results are generalized to the pyramids consisting of the generalized coefficients defined by a strong divisibility sequence.

1. INTRODUCTION

In our previous paper [6], we showed the reason why there exists the dual correspondence between sets in Pascal's triangle and one's in the modified Pascal triangle concerning GCD and LCM for which many examples are given in [1] and [2]. In this note we will extend the results to the case of the m-dimensional generalized Pascal pyramid and the m-dimensional generalized modified Pascal pyramid. We have shown in our previous paper [3] and [4] many examples in which a GCD equality for a configuration C in Pascal's pyramid $\Delta(P)$ corresponds to an LCM equality for a configuration C', which is symmetric to C with respect to a point, in modified Pascal pyramid $\Delta(M)$, and an LCM equality for a configuration C in Pascal's pyramid $\Delta(P)$ corresponds to a GCD equality for a configuration C' in modified Pascal pyramid $\Delta(M)$ (the definitions will be given in the next section).

The purpose of this paper is to clarify the reason why such a phenomenon occurs between these pyramidal arrays of numbers by showing a p-adic complementary relation of m-nomial coefficients and modified m-nomial coefficients.

2. DEFINITIONS, NOTATIONS AND CLARIFICATIONS

We denote the value of m-nomial coefficients as

$$\binom{n}{r_1 r_2 \dots r_m} = \frac{n!}{r_1!r_2!\dots r_m!},$$

where $n = r_1 + r_2 + \cdots + r_m$. The m-dimensional Pascal's pyramid, which we denote by $\Delta(P)$, is the m-dimensional pyramidal array of m-nomial coefficients. We call

$$\left\{ \begin{matrix} n \\ r_1 r_2 \dots r_m \end{matrix} \right\} = \frac{(n+m-1)!}{r_1! r_2! \dots r_m!},$$

where $n = r_1 + r_2 + \cdots + r_m$, the modified m-nomial coefficients, and we refer to a similar pyramidal array of these coefficients as the m-dimensional modified Pascal pyramid, which we denote by $\Delta(M)$.

Let the symbols $\left(\begin{matrix} n \\ r_1 r_2 \dots r_m \end{matrix} \right)$ and $\left\{ \begin{matrix} n \\ r_1 r_2 \dots r_m \end{matrix} \right\}$ represent both their values and positions in $\Delta(P)$ and $\Delta(M)$, respectively. Since the position of a point is determined by $r_1, r_2, \dots, r_m$, in both $\Delta(P)$ and $\Delta(M)$, we can represent it by $(r_1, r_2, \dots, r_m)$, which we refer to the coordinates of the point. We overlap $\Delta(P)$ and $\Delta(M)$ in m-dimensional space in such a way that any two points taken out of each pyramid by one coincide if they have the same coordinates. When we consider symbols or concepts common to $\Delta(P)$ and $\Delta(M)$, and do not have to distinguish them, we sometimes use the symbol $\Delta(T)$ to represent both $\Delta(P)$ and $\Delta(M)$ in order to avoid repetitions. It is assumed throughout the arguments that $r_1, r_2, \dots, r_m$ are nonnegative integers satisfying $n = r_1 + r_2 + \cdots + r_m$. A nonempty finite subset C of $\Delta(T)$ is called a configuration in $\Delta(T)$. We introduce an equivalence relation to the set of all the configurations in $\Delta(T)$ such that two configurations in $\Delta(T)$ are equivalent to each other if and only if one is obtained by a parallel translation of the other. Then an equivalence class of the set of all configurations in $\Delta(T)$ by this equivalence relation is called a translatable configuration in $\Delta(T)$. Unless otherwise stated, we simply call it a configuration C even if it is actually referring the translatable configuration to which the configuration C belongs. There will not be danger of misinterpretation since we are discussing only the GCD and LCM properties which hold on C independent of the location of C in $\Delta(T)$.

Let S_1 and S_2 be two nonempty finite subsets of $\Delta(T)$, where $S_1 \cap S_2 = \emptyset$ is not claimed. We define a configuration C in $\Delta(T)$ by $C = S_1 \cup S_2$. If the equality

$$\gcd(S_1) = \gcd(S_2) \tag{1}$$

holds independent of the location of C in $\Delta(T)$, we call (1) a GCD equality in $\Delta(T)$. In the same manner, if

$$\operatorname{lcm}(S_1) = \operatorname{lcm}(S_2) \tag{2}$$

holds instead of (1), we call (2) a LCM equality in $\Delta(T)$.

Let us fix a point X in $\Delta(T)$. If X is a midpoint of segment AA', we say that two points A and A' are symmetric with respect to X. Two configurations C and C' are said to be symmetric with respect to X if there is a one to one correspondence σ from points of C to points of C', such that all the corresponding pairs are symmetric with respect to X, and then σ is called a symmetric transformation.

Notice that each transformation operates on C, not on $\Delta(T)$, and therefore, when we consider the case in which a symmetric transformation operates on a translatable configuration C, we do not have to locate the center of symmetry since change of the center of symmetry only causes a parallel transformation of the resulting C' which is unchanged as a translatable configuration.

3. p-ADIC COMPLEMENTARY THEOREM BETWEEN m-NOMIAL COEFFICIENTS AND MODIFIED m-NOMIAL COEFFICIENTS

Now we fix a prime p and integers $m \geq 2$ and $e \geq 1$, and let $r_1, r_2, \ldots, r_m, r'_1, r'_2, \ldots, r'_m, n = r_1 + r_2 + \cdots + r_m, n' = r'_1 + r'_2 + \cdots + r'_m$ be nonnegative integers. The additive p-adic valuation of an integer y, denoted by $\beta = v_p(y)$, is the largest integer β such that p^β divides y. We assume that the conditions

$$n + n' = mp^e - m, \quad r_1 + r'_1 = r_2 + r'_2 = \cdots = r_m + r'_m = p^e - 1 \tag{3}$$

are satisfied. Then we put

$$v_p(n, n') = v_p \binom{n}{r_1 r_2 \ldots r_m} + v_p \left\{ \begin{matrix} n' \\ r'_1 r'_2 \ldots r'_m \end{matrix} \right\},$$

since the right hand side depends only on n and n' and is independent of $r_1, r_2, \ldots, r_m$ and $r'_1, r'_2, \ldots, r'_m$, as we will see in the proof of the following theorem.

Theorem 1: (p-adic complementary theorem) If the conditions $(k-1)p^e \leq n \leq kp^e - 1$ or $(m-k)p^e - m + 1 \leq n' \leq (m-k+1)p^e - m$ are satisfied for a positive integer k, then the value of $v_p(n, n')$ is unchanged.

Proof: If $r + r' = p^e - 1$, then $v_p(r') = v_p(r+1)$, and so we have

$$\begin{aligned} v_p(r!r'!) &= v_p(r!) + v_p(r'!) = v_p(r!) + v_p((r+1)(r+2)\ldots(r+r')) \\ &= v_p((r+r')!) = v_p((p^e-1)!). \end{aligned}$$

If $n > 0$, since $v_p(n' + m + h) = v_p(n - h)$ for $h = 0, 1, \ldots, n - (k-1)p^e - 1$ by the conditions,

$$\begin{aligned} v_p(n!) &= v_p(n(n-1)\cdots((k-1)p^e + 1)) + v_p(((k-1)p^e)!) \\ &= v_p((n'+m)(n'+m+1)\cdots(n'+m+n-(k-1)p^e - 1) \\ &\quad + v_p(((k-1)p^e)!) \\ &= v_p((n'+m)(n'+m+1)\cdots((m-k+1)p^e - 1)) + v_p(((k-1)p^e)!) \\ v_p(n!(n'+m-1)!) &= v_p(n!) + v_p((n'+m-1)!) \\ &= v_p((n'+m)(n'+m+1)\cdots((m-k+1)p^e - 1)) + v_p(((k-1)p^e)!) \\ &\quad + v_p((n'+m-1)!) \\ &= v_p(((k-1)p^e!) + v_p((m-k+1)p^e - 1)!). \end{aligned}$$

The last equality is also trivially valid for $n = 0$ (then $k = 1$). Thus

$$\begin{aligned} v_p(n, n') &= v_p(n!(n'+m-1)!/r_1!r'_1!r_2!r'_2!\ldots r_m!r'_m!) \\ &= v_p(((k-1)p^e)!((m-k+1)p^e - 1)!/((p^e-1)!)^m), \end{aligned}$$

which depends only on p, m, k and e.

Remark: If we do not restrict the range of n, $v_p(n, n')$ does not always keep the same value. For example, in the case $c = 2$, $m = 5$ and $e = 1$, the values of $v_p(n, n')$ are as follows.

n	0	1	2	3	4
$v_p(n, n')$	7	7	5	5	6

Theorem 2: Let $m = kp^f$, where $f \geq 0$ and k is not divisible by p. Then $v_p(n, n')$ takes the same value for any $r_1, r_2, \ldots, r_m, r'_1, r'_2, \ldots, r'_m$, $n = r_1 + r_2 + \cdots + r_m$ and $n' = r'_1 + r'_2 + \cdots + r'_m$, satisfying (3), if and only if $k < p$.

Proof: If $k < p, n < mp^e < p^{f+1}p^e = p^{e+f+1}$ so that $v_p(n) \leq e + f$ for any n such that $n \leq mp^e - m$. Thus $v_p(n' + m + h) = v_p(n - h)$ for $h = 0, 1, \ldots, n - 1$, and we have

$$v_p(n!(n' + m - 1)!) = v_p((n + n' + m - 1)!) = v_p((mp^e - 1)!).$$

If $k > p$ on the contrary, for $n = p^{f+1+e} < mp^e$,

$$v_p(n) = f + 1 + e, \text{ while } v_p(n' + m) = v_p(mp^e - n) = v_p((k - p)p^{f+e}) = f + e.$$

Since $v_p(n' + m + h) = v_p(n - h)$ for $h = 1, 2, \ldots, n - 1$, we have

$$v_p(n, n') = v_p(n - 1, n' + 1) + 1$$

to complete the proof.

4. GCD-LCM DUALITY BETWEEN m-DIMENSIONAL PASCAL'S PYRAMID AND THE MODIFIED PASCAL PYRAMID

As an application of the p-adic complementary theorem between the m-nomial coefficients and the modified m-nomial coefficients which was established in the previous section, we now prove a duality between a configuration C in m-dimensional Pascal's pyramid $\Delta(P)$ and a configuration C' in m-dimensional modified Pascal pyramid $\Delta(M)$ concerning the GCD and the LCM.

Theorem 3: Let $C = S_1 \cup S_2$ be a configuration in $\Delta(P)$ consisting of two nonempty finite subsets S_1 and S_2, which corresponds to a configuration $C' = \sigma(C) = S'_1 \cup S'_2$ in $\Delta(M)$ by a symmetric transformation σ with respect to a point X in such a manner that $\sigma(S_1) = S'_1$ and $\sigma(S_2) = S'_2$. Then the GCD equality

$$\gcd(S_1) = \gcd(S_2) \tag{4}$$

holds wherever C is located in $\Delta(P)$, if and only if the LCM equality

$$\operatorname{lcm}(S'_1) = \operatorname{lcm}(S'_2) \tag{5}$$

holds wherever $C' = \sigma(C)$ is located in $\Delta(M)$. Similarly, the LCM equality for C holds in $\Delta(P)$ if and only if the corresponding GCD equality for C' holds in $\Delta(M)$.

Proof: First, we assume that the GCD equality (4) holds independent of the location of C in $\Delta(P)$. Then wherever C is located in $\Delta(P)$, we have

$$\min\{v_p(S_1)\} = \min\{v_p(S_2)\}, \tag{6}$$

where $\min\{v_p(S_i)\}$ denotes $\min\{v_p(A)|A \in S_i\}$ for $i = 1$ and 2.

Let p be an arbitrary, but fixed, prime. Now we take the point X whose coordinates are given by $r_1 = r_2 = \cdots = r_m = (p^e - 1)/2$ as the center of the symmetric transformation σ. Notice that if $p = 2$, X does not correspond to an entry of $\Delta(T)$. Given a location of C' in $\Delta(M)$, we take e to be sufficiently large so that

$$p^e > \max\{n'|n' \text{ corresponds to a point } A' \text{ in } C'\} + m.$$

Then the configuration C corresponding to C' by σ is contained in $\Delta(P)$, and each point $A \in C$ satisfies the condition of Theorem 1 for $k = m$. Therefore, $v_p(A) + v_p(A')$ takes the same value, which we denote by $v_p(C, C')$, for every $A \in C$ and the corresponding $A' = \sigma(A) \in C'$, so that

$$\min\{v_p(S)\} + \max\{v_p(S')\} = v_p(C, C') = \text{ a constant} \tag{7}$$

for any $S \subset C$ and corresponding $S' \subset C'$.

Since we assume the GCD equality (4) in $\Delta(P)$, equality (6) holds so that, using (7), we have

$$\max\{v_p(S_1')\} = \max\{v_p(S_2')\} \text{ for all primes } p, \tag{8}$$

which is equivalent to (5). Similarly, we can prove that if (5) holds independent of the location of C' in $\Delta(M)$, then (4) holds independent of the location of C in $\Delta(P)$.

If we exchange min and max in (6), (7) and (8), then gcd and lcm in (4) and (5) must be exchanged.

5. THE CASE OF GENERALIZED m-NOMIAL COEFFICIENTS AND GENERALIZED MODIFIED m-NOMIAL COEFFICIENTS

A sequence of positive integers $A = \{a_n\} = \{a_1, a_2, a_3, \cdots\}$ is called a strong divisibility sequence if

$$(a_k, a_h) = a_{(k,h)}$$

for every $k, h = 1, 2, 3, \cdots$, where (a_k, a_h) and (k, h) are the greatest common divisors of two numbers. Without loss of generality, we can assume here that $a_1 = 1$ as we stated in [5]. The sequence of natural numbers $N = \{1, 2, 3, \ldots\}$ and the sequence of Fibonacci numbers $F = \{F_1, F_2, F_3, \ldots\}$ are examples of strong divisibility sequences. Let p^e, where $e > 0$ be a prime power. The rank of apparition of p^e in $A = \{a_n\}$, the smallest u such that $p^e|a_u$ is denoted by $\rho(p^e)$. In our previous paper [5], we showed that the sequence $\{k_n\}$ defined by k_n $=\min\{e, v_p(a_n)\}$ is periodic with period $u = \rho(p^e)$ and is symmetric in the interval $0 < n < u$.

For any strong divisibility sequence $A = \{a_n\}$, if we generalize the m-nomial coefficients and the modified m-nomial coefficients by replacing $r_1, r_2, \cdots, r_m$ and n in section 2 with $a_{r_1}, a_{r_2}, \cdots, a_{r_m}$ and a_n, respectively, then we have A m-nomial coefficents

$$\binom{n}{r_1 r_2 \cdots r_m}_A = \frac{\prod a_n}{\prod a_{r_1} \prod a_{r_2} \cdots \prod a_{r_m}},$$

and modified A m-nomial coefficients

$$\left\{ \begin{matrix} n \\ r_1 r_2 \cdots r_m \end{matrix} \right\}_A = \frac{\prod a_{n+m-1}}{\prod a_{r_1} \prod a_{r_2} \cdots \prod a_{r_m}},$$

where $n = r_1 + r_2 + \cdots + r_m$ and the symbol $\prod a_r$ is defined by $\prod a_r = a_1 a_2 \cdots a_r$ for $r \in N$.

For the generalized Pascal Pyramid and the generalized modified Pascal Pyramid consisting of these coefficients, we use the same symbols as we defined for original ones in section 2, or sometimes put sub A to distinguish from the original case.

Let $A = \{a_n\}$ be a strong divisibility sequence, and p a fixed prime. We assume that $n = r_1 + r_2 + \cdots + r_m$ and $n' = r'_1 + r'_2 + \cdots + r'_m$ as above and the conditions

$$n + n' = mu - m, \quad r_1 + r'_1 = r_2 + r'_2 = \cdots = r_m + r'_m = u - 1 \tag{9}$$

are satisfied, where $u = \rho(p^e)$ for a prime power p^e. This means geometrically that a point $(r_1, r_2, \ldots, r_m)$ in $\Delta(P)_A$ and a point $(r'_1, r'_2, \ldots, r'_m)$ in $\Delta(M)_A$ are symmetric with respect to a center X whose coordinates are given by $r_1 = r_2 = \cdots = r_m = (u-1)/2$. Then we put

$$v_p(n, n')_A = v_p \left(\begin{matrix} n \\ r_1 r_2 \ldots r_m \end{matrix} \right)_A + v_p \left\{ \begin{matrix} n' \\ r'_1 r'_2 \ldots r'_m \end{matrix} \right\}_A .$$

Theorem 4: (p-adic complementary theorem for the generalized version)

If $\{v_p(a_n)\}$ is bounded, then $v_p(n, n')_A$ is a constant.

If $\{v_p(a_n)\}$ is unbounded, then the value of $v_p(n, n')_A$ is unchanged in the region where $(k-1)u \le n \le ku - 1$ or $(m-k)u - m + 1 \le n' \le (m-k+1)u - m$ are satisfied for a positive integer k.

Proof: If $r + r' = u - 1$, then $v_p(r') = v_p(u - (r+1)) = v_p(r+1)$, and so we have $v_p(\prod a_r \prod a_{r'}) = v_p(\prod a_r) + v_p(\prod a_{a'_r}) = +v_p(\prod a_r) + v_p(a_{r+1} a_{r+2} \ldots a_{r+r'}) = v_p(\prod a_{r+r'}) = v_p(\prod a_{u-1})$.

First, we assume that $\{v_p(a_n)\}$ is bounded. Let $\max\{v_p(a_n)\} = e$ and put $u = \rho(p^e)$. Then $\{v_p(a_n)\}$ is periodic with period u and symmetric in the interval $1 \le n \le u - 1$. Since $n + n' + m - 1 = mu - 1$, we have $n' + m - k = mu - (n + k)$ for $k = 1, 2, \cdots, n' + m - 1$, and so,

$$\begin{aligned} v_p \left(\prod a_n \prod a_{n'+m-1} \right) &= v_p \left(\prod a_n \right) + v_p \left(\prod a_{n'+m-1} \right) \\ &= v_p \left(\prod a_n \right) + v_p \left(a_{n+1} a_{n+2} \cdots a_{mu-1} \right) = v_p \left(\prod a_{mu-1} \right). \end{aligned}$$

Summarizing above results, we have

$$v_p (n, n')_A = v_p \left(\prod a_{mu-1} \right) - m v_p \left(\prod a_{u-1} \right),$$

which shows that $v_p(n, n')_A$ is independent of n and n' as was desired.

In the second case, where $\{v_p(a_n)\}$ is unbounded, $v_p(n, n')_A$ is not always a constant in the whole region. If we restrict the region, however, we can prove that $v_p(n, n')_A$ is unchanged in $(k-1)u \leq n \leq ku-1$ or $(m-k)u-m+1 \leq n' \leq (m-k+1)u-m$ for a positive integer k. The proof is similar to that of Theorem 1 or the first part of this theorem, and will be omitted.

This theorem also leads the next one in the same way as we got Theorem 3.

Theorem 5: Let $C = S_1 \cup S_2$ be a configuration in $\Delta(P)_A$ consisting of two nonempty finite subsets S_1 and S_2, which corresponds to a configuration $C' = S_1' \cup S_2'$ in $\Delta(M)_A$ by a symmetric transformation σ with respect to a point X in a manner $\sigma(S_1) = S_1'$ and $\sigma(S_2) = S_2'$. Then the GCD equality

$$\gcd(S_1) = \gcd(S_2)$$

holds wherever C is located in $\Delta(P)_A$, if and only if the LCM equality

$$\operatorname{lcm}(S_1') = \operatorname{lcm}(S_2')$$

holds wherever $C' = \sigma(C)$ is located in $\Delta(M)_A$. Similarly, the LCM equality for C holds in $\Delta(P)_A$ if and only if the corresponding GCD equality for C' holds in $\Delta(M)_A$.

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AMS Classification Numbers: 11A99, 11B65, 05A10

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