DISTRIBUTION OF THE FIBONACCI NUMBERS MODULO 3^k

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1. INTRODUCTION

Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, denote the sequence of Fibonacci numbers. For an integer $m \geq 2$, we shall consider Fibonacci numbers in $\mathbb{Z}_m$ throughout this paper. It is known that the sequence $\{F_n \pmod{m}\}_{n\geq 0}$ is periodic [8]. Let $\pi(m)$ denote the (shortest) period of this sequence. There are some known results on $\pi(m)$ [2, 6, 7, 8].

Theorem 1.1: [8] If $\pi(p) \neq \pi(p^2)$, then $\pi(p^k) = p^{k-1}\pi(p)$ for each integer $k \geq 1$ and prime p. Also if t is the largest integer with $\pi(p^t) = \pi(p)$, then $\pi(p^k) = p^{k-t}\pi(p)$ for $k > t$.

For any modulus $m \geq 2$ and residue $b \pmod{m}$ (we always assume $1 \leq b \leq m$), denote by $\nu(m, b)$ the frequency of b as a residue in one period of the sequence $\{F_n \pmod{m}\}$. It was proved that $\nu(5^k, b) = 4$ for each $b \pmod{5^k}$ and each $k \geq 1$ by Niederreiter in 1972 [7]. Jacobson determined $\nu(2^k, b)$ for $k \geq 1$ and $\nu(2^k5^j, b)$ for $k \geq 5$ and $j \geq 0$ in 1992 [6]. Some other results in this area can be found in [4, 5].

In this paper we shall partially describe the number $\nu(3^k, b)$ for $k \geq 1$.

Example 1.1: A period of $F_n \pmod{27}$ is listed below:

F_{8x+y} ↘	1	2	3	4	5	6	7	8	← y
0	1	1	2	3	5	8	13	21	
1	7	1	8	9	17	26	16	15	
2	4	19	23	15	11	26	10	9	
3	19	1	20	21	14	8	22	3	
4	25	1	26	0	26	26	25	24	
5	22	19	14	6	20	26	19	18	
6	10	1	11	12	23	8	4	12	
7	16	1	17	18	8	26	7	6	
8	13	19	5	24	2	26	1	0	
x ↑									

Table 1: A period of the Fibonacci numbers $F_{8x+y} \pmod{27}$.

So $\nu(27, 1) = \nu(27, 26) = 8$, $\nu(27, 8) = \nu(27, 19) = 5$ and $\nu(27, b) = 2$ for $b \neq 1, 8, 19, 26$. □

2. SOME KNOWN RESULTS

In Section 4, we shall consider the frequency of each residue $b \pmod{3^k}$ in one period of the sequence $\{F_n \pmod{3^k}\}$. Before considering this problem we list some well-known identities in this section.

The Fibonacci sequence is defined for all integer values of the index n. So we have

$$F_{-n} = (-1)^{n+1} F_n; \tag{1}$$

$$F_{n+m} = F_{m-1}F_n + F_m F_{n+1}; \tag{2}$$

$$F_{kn+r} = \sum_{h=0}^{k} \binom{k}{h} F_n^h F_{n-1}^{k-h} F_{r+h}, \text{ for } k \geq 0; \tag{3}$$

$$F_{kn} = F_n \sum_{h=1}^{k} \binom{k}{h} F_n^{h-1} F_{n-1}^{k-h} F_h, \text{ for } k \geq 0; \tag{4}$$

Remark: The proof of (1) can be found in [1]. (2) was mentioned as a known result in the proof of [8, Theorem 3]. It is called the addition formula. (3) was mentioned in [2] as a known result. These two identities can be proved by induction. (4) follows from the fact $F_0 = 0$ and (3).

¿From (3), (4) and the fact $F_{-1} = F_1 = F_2 = 1$, $F_0 = 0$ and $F_3 = 2$, we have

$$F_{3n-1} = (F_{n-1})^3 + 3(F_n)^2 F_{n-1} + (F_n)^3, \tag{5}$$

$$F_{3n} = F_n \left[3(F_{n-1})^2 + 3F_n F_{n-1} + 2(F_n)^2\right]. \tag{6}$$

Let $\alpha(m^k)$ be the first index $\alpha > 0$ such that $F_\alpha \equiv 0 \pmod{m^k}$. Let $\beta(m^k)$ be the largest integer β such that $F_{\alpha(m^k)} \equiv 0 \pmod{m^\beta}$, i.e., $\beta(m^k)$ is the largest exponent β such that m^β divides $F_{\alpha(m^k)}$. It is usually written as $m^{\beta(m^k)} \| F_{\alpha(m^k)}$ in number theory. Note that, by using the fact that the g.c.d.$(F_\alpha, F_{\alpha-1}) = 1$ and (3) we have $\alpha(m)$ is a factor of $\pi(m)$ for $m \geq 2$ (the reader also may wish to see [2]).

Theorem 2.1: [2] If p is an odd prime and $k \geq \beta(p)$, then $\alpha(p^k) = p^{k-\beta(p)}\alpha(p)$ and $\beta(p^k) = k$.

Example 2.1: $\{F_n \pmod 3\}_{n\geq 0} = \{0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \ldots\}$. Thus we have $\pi(3) = 8$ and $\alpha(3) = 4$. Since $F_4 = 3$, $\beta(3) = 1$. By Theorem 2.1, $\alpha(3^k) = 3^{k-1}\alpha(3) = 4 \cdot 3^{k-1}$ and $\beta(3^k) = k$ for $k \geq 1$. This means that $3^k \| F_{4 \cdot 3^{k-1}}$ for $k \geq 1$. □

It is easy to check that $\pi(3) = 8$ and $\pi(3^2) = 24$. Applying Theorem 1.1 we have

$$\pi(3^k) = 8 \cdot 3^{k-1} \text{ for } k \geq 1.$$

From Example 2.1, we have

$$3^k \| F_{\pi(3^k)/2} \text{ for } k \geq 1. \tag{7}$$

3. SOME USEFUL IDENTITIES OF FIBONACCI NUMBERS MODULO 3^k

In this section, we show some identities of Fibonacci numbers modulo 3^k which will be used in Section 4.

Lemma 3.1: *For* $k \geq 4$, $F_{\pi(3^k)/9-1} \equiv 7 \cdot 3^{k-2} + 1 \pmod{3^k}$ *and* $F_{\pi(3^k)/9} \equiv 4 \cdot 3^{k-2} \pmod{3^k}$.

Proof: Note that $\pi(3^k) = 8 \cdot 3^{k-1}$. We prove this lemma by induction on k. When $k = 4$, we have $F_{23} = 28657 \equiv 64 \equiv 7 \cdot 3^2 + 1 \pmod{3^4}$ and $F_{24} = 46368 \equiv 36 \equiv 4 \cdot 3^2 \pmod{3^4}$.

Suppose the lemma is true for some $k \geq 4$. Since $2k - 3 \geq k + 1$ and $F_{8\cdot 3^{k-3}} \equiv 0 \pmod 3$,

$$3\left(F_{8\cdot 3^{k-3}}\right)^2 \equiv 0 \pmod{3^{k+1}} \tag{8}$$

$$\left(F_{8\cdot 3^{k-3}}\right)^3 \equiv 0 \pmod{3^{k+1}} \tag{9}$$

$$\text{and } \left(F_{8\cdot 3^{k-3}-1}\right)^3 \equiv \left(7\cdot 3^{k-2}+1\right)^3 \equiv 7\cdot 3^{k-1}+1 \pmod{3^{k+1}}. \tag{10}$$

By putting $n = 8 \cdot 3^{k-3}$ into (5) and (6), using (8), (9), (10) and the induction assumption, we have

$$F_{8\cdot 3^{k-2}-1} \equiv \left(F_{8\cdot 3^{k-3}-1}\right)^3 \equiv 7\cdot 3^{k-1}+1 \pmod{3^{k+1}},$$

$$\begin{aligned} F_{8\cdot 3^{k-2}} &\equiv 3F_{8\cdot 3^{k-3}}\left(F_{8\cdot 3^{k-3}-1}\right)^2 \\ &\equiv 3(4\cdot 3^{k-2}+3^k u)(7\cdot 3^{k-2}+1+3^k v)^2 \qquad \text{for some } u, v \in \mathbb{Z} \\ &\equiv 4\cdot 3^{k-1}[3^{2k-4}(7+9v)^2 + 2\cdot 3^{k-2}(7+9v)+1] \equiv 4\cdot 3^{k-1} \pmod{3^{k+1}}. \end{aligned}$$

This completes the proof. □

Corollary 3.2: *For* $k \geq 2$, $F_{\frac{\pi}{3}-1} \equiv 3^{k-1}+1 \pmod{3^k}$ *and* $F_{\frac{\pi}{3}} \equiv 3^{k-1} \pmod{3^k}$, where $\pi = \pi(3^k)$.

Proof: Suppose $k = 2$. $F_7 = 13 \equiv 4 \pmod{3^2}$ and $F_8 = 21 \equiv 3 \pmod{3^2}$. Suppose $k = 3$. By the proof of Lemma 3.1 we have $F_{23} \equiv 7 \cdot 3^2 + 1 \pmod{3^4}$ and $F_{24} \equiv 4 \cdot 3^2 \pmod{3^4}$. This implies $F_{23} \equiv 3^2 + 1 \pmod{3^3}$ and $F_{24} \equiv 3^2 \pmod{3^3}$. Suppose $k \geq 4$. By (5), (8), (9) and (10) we have

$$\begin{aligned} F_{\frac{\pi}{3}-1} &= \left(F_{\frac{\pi}{9}-1}\right)^3 + 3\left(F_{\frac{\pi}{9}}\right)^2 F_{\frac{\pi}{9}-1} + \left(F_{\frac{\pi}{9}}\right)^3 \\ &\equiv 7\cdot 3^{k-1}+1 \pmod{3^{k+1}} \\ &\equiv 3^{k-1}+1 \pmod{3^k}. \end{aligned}$$

Similarly, by (6) (8), (9) and (10) we have

$$\begin{aligned} F_{\frac{\pi}{3}} &\equiv 3F_{\frac{\pi}{9}}\left(F_{\frac{\pi}{9}-1}\right)^2 \pmod{3^{k+1}} \\ &\equiv 3\cdot 4\cdot 3^{k-2}(7\cdot 3^{k-2}+1)^2 \pmod{3^k} \\ &\equiv 4\cdot 3^{k-1} \equiv 3^{k-1} \pmod{3^k}. \end{aligned}$$

This completes the proof. □

Proposition 3.3 can be proved like Lemma 3.1 was proved. However, we will provide another proof.

Proposition 3.3: *For* $k \geq 1$, $F_{\frac{\pi}{2}-1} = F_{\alpha(3^k)-1} \equiv -1 \pmod{3^k}$, *where* $\pi = \pi(3^k)$.

Proof: By (2) we have $F_{\pi-1} = \left(F_{\frac{\pi}{2}-1}\right)^2 + \left(F_{\frac{\pi}{2}}\right)^2$. By (7) we have $\left(F_{\frac{\pi}{2}-1}\right)^2 \equiv 1 \pmod{3^k}$. By the definition of π and together with (7), $F_{\frac{\pi}{2}-1} \not\equiv 1 \pmod{3^k}$. Since the multiplication group of units of $\mathbb{Z}_{3^k}$ is cyclic (see [3, Theorem 4.19]), $F_{\frac{\pi}{2}-1} \equiv -1 \pmod{3^k}$. □

Corollary 3.4: *For* $k \geq 2$, $F_{n+\frac{\pi}{2}} \equiv -F_n \pmod{3^k}$.

Proof: By (2) we have $F_{n+\frac{\pi}{2}} = F_{\frac{\pi}{2}-1}F_n + F_{\frac{\pi}{2}}F_{n+1}$. By Proposition 3.3 and (7) we have $F_{n+\frac{\pi}{2}} \equiv -F_n \pmod{3^k}$. □

Thus, for each b and each n such that $F_n \equiv b \pmod{3^k}$ we have $F_{n+\frac{\pi}{2}} \equiv -b \pmod{3^k}$. Thus the frequency of $b \pmod{3^k}$ and $-b \pmod{3^k}$ are equal. That is, $\nu(3^k, b) = \nu(3^k, 3^k - b)$.

4. FREQUENCIES OF FIBONACCI NUMBERS MODULO 3^k

In this section, we shall compute some values of $\nu(3^k, b)$ for $k \geq 1$.

Lemma 4.1 *For* $k \geq 2$, *we have* $F_{n+\frac{\pi}{3}} \equiv \left\{\begin{array}{ll} F_n & \text{if } n \equiv 2,\ 6 \pmod 8 \\ F_n + 3^{k-1} & \text{if } n \equiv 0,\ 5,\ 7 \pmod 8 \\ F_n + 2\cdot 3^{k-1} & \text{if } n \equiv 1,\ 3,\ 4 \pmod 8 \end{array}\right\}$ $\pmod{3^k}$, *where* $\pi = \pi(3^k)$.

Proof: By (2) and Corollary 3.2, we have

$$F_{n+\frac{\pi}{3}} = F_nF_{\frac{\pi}{3}-1} + F_{n+1}F_{\frac{\pi}{3}} \equiv (3^{k-1}+1)F_n + 3^{k-1}F_{n+1} \equiv F_n + 3^{k-1}F_{n+2} \pmod{3^k}. \quad (11)$$

Since $\pi(3) = 8$ and $\{F_{n+2} \pmod 3\}_{n\geq 0} = \{1, 2, 0, 2, 2, 1, 0, 1, \ldots\}$, we obtain the lemma. □

Lemma 4.2: *For* $k \geq 4$, *we have* $F_{n+\frac{\pi}{9}} \equiv \left\{\begin{array}{ll} F_n & \textit{if } n \equiv 6,\ 18 \pmod{24} \\ F_n + 3^{k-1} & \textit{if } n \equiv 10,\ 14 \pmod{24} \\ F_n + 2\cdot 3^{k-1} & \textit{if } n \equiv 2,\ 22 \pmod{24} \end{array}\right\}$ $\pmod{3^k}$, *where* $\pi = \pi(3^k)$

Proof: By (2) and Lemma 3.1, we have

$$F_{n+\frac{\pi}{9}} = F_nF_{\frac{\pi}{9}-1} + F_{n+1}F_{\frac{\pi}{9}} \equiv F_n + 3^{k-2}(7F_n + 4F_{n+1}) \pmod{3^k}.$$

Let $U_n = 7F_n + 4F_{n+1}$. Since $\pi(9) = 24$ and $U_n \equiv 6, 0, 3, 3, 0, 6 \pmod 9$ when $n \equiv 2, 6, 10, 14, 18, 22 \pmod{24}$, respectively, we have the lemma. □

For each b, $1 \leq b \leq 27$, we let the number $\omega(3^k, b) = |\{n \mid F_n \equiv b \pmod{27},\ 1 \leq n \leq \pi(3^k)\}|$. This means that

$$\omega(3^k, b) = \sum_{\substack{1\leq x\leq 3^k \\ x\equiv b \pmod{27}}} \nu(3^k, x).$$

Let A be a set of one period of the sequence $\{F_n \pmod{3^k}\}$, where $k \geq 3$. Since $\pi(3^k) = 3^{k-3}\pi(27)$, after taking modulo 27 for each element of A, the set A becomes 3^{k-3} copies of a period of the sequence $\{F_n \pmod{27}\}$. Thus by Example 1.1 we have the following lemma.

Lemma 4.3: *For* $k \geq 3$, $\omega(3^k, b) = \begin{cases} 8\cdot 3^{k-3} & \textit{if } b = 1,\ 26 \\ 5\cdot 3^{k-3} & \textit{if } b = 8,\ 19\ . \\ 2\cdot 3^{k-3} & \textit{otherwise.} \end{cases}$

Lemma 4.4: *Let $k \geq 1$. Suppose $1 \leq n \leq \pi(3^k)$ with $n \not\equiv 2, 6 \pmod 8$. If $F_n \equiv b \pmod{3^k}$, then there is a number $n' \in \{n, n+\pi(3^k), n+2\pi(3^k)\}$ such that $F_{n'} \equiv b \pmod{3^{k+1}}$. Moreover, two sets $\{F_n,\ F_{n+\pi(3^k)},\ F_{n+2\pi(3^k)}\}$ and $\{b,\ b+3^k,\ b+2\cdot 3^k\}$ are equal in $\mathbb{Z}_{3^{k+1}}$.* Note that $n \equiv n' \pmod 8$.

Proof: It is straightforward to check that the lemma holds for $k = 1$.

Now we assume $k \geq 2$. Suppose $F_n \equiv b' \pmod{3^{k+1}}$. Then $b' \equiv b + 3^k c \pmod{3^{k+1}}$, for some c with $0 \leq c \leq 2$.

Now $\frac{\pi(3^{k+1})}{3} = \pi(3^k)$, so by Lemma 4.1 we have

$$F_{n+\pi(3^k)} = F_{n+\frac{\pi(3^{k+1})}{3}} \equiv \left\{ \begin{array}{ll} F_n + 3^k & n \equiv 0,5,7 \pmod 8 \\ F_n + 2\cdot 3^k & n \equiv 1,3,4 \pmod 8 \end{array} \right\} \pmod{3^{k+1}}$$

$$\equiv \left\{ \begin{array}{ll} b' + 3^k & n \equiv 0,5,7 \pmod 8 \\ b' + 2\cdot 3^k & n \equiv 1,3,4 \pmod 8 \end{array} \right\} \pmod{3^{k+1}}$$

$$\equiv \left\{ \begin{array}{ll} b + 3^k(c+1) & n \equiv 0,5,7 \pmod 8 \\ b + 3^k(c+2) & n \equiv 1,3,4 \pmod 8 \end{array} \right\} \pmod{3^{k+1}}.$$

Since $\pi(3^k) \equiv 0 \pmod 8$, $n \equiv n + \pi(3^k) \equiv n + 2\pi(3^k) \pmod 8$. So we have $\{F_n,\ F_{n+\pi(3^k)},\ F_{n+2\pi(3^k)}\} = \{b,\ b+3^k,\ b+2\cdot 3^k\}$ in $\mathbb{Z}_{3^{k+1}}$. This completes the proof. □

Lemma 4.5: *Let $k \geq 3$. Suppose $1 \leq n \leq \pi(3^k)$ with $n \equiv 2, 10, 14, 22 \pmod{24}$. If $F_n \equiv b \pmod{3^k}$, then there is a number $n' \in \{n, n+\frac{\pi(3^k)}{3}, n+\frac{2\pi(3^k)}{3}\}$ such that $F_{n'} \equiv b \pmod{3^{k+1}}$. Moreover, two sets $\left\{F_n,\ F_{n+\frac{\pi(3^k)}{3}},\ F_{n+\frac{2\pi(3^k)}{3}}\right\}$ and $\{b,\ b+3^k,\ b+2\cdot 3^k\}$ are equal in $\mathbb{Z}_{3^{k+1}}$.* Note that $n \equiv n' \pmod{24}$.

Proof: Suppose $F_n \equiv b' \pmod{3^{k+1}}$. Then $b' \equiv b + 3^k c \pmod{3^{k+1}}$, for some c with $0 \leq c \leq 2$.

Similar to the proof of Lemma 4.4, now $\frac{\pi(3^{k+1})}{9} = \frac{\pi(3^k)}{3}$, so by Lemma 4.2 we have

$$F_{n+\frac{\pi(3^k)}{3}} = F_{n+\frac{\pi(3^{k+1})}{9}} \equiv \left\{ \begin{array}{ll} F_n + 3^k & n \equiv 10,14 \pmod{24} \\ F_n + 2\cdot 3^k & n \equiv 2,22 \pmod{24} \end{array} \right\} \pmod{3^{k+1}}$$

$$\equiv \left\{ \begin{array}{ll} b + 3^k(c+1) & n \equiv 10,14 \pmod{24} \\ b + 3^k(c+2) & n \equiv 2,22 \pmod{24} \end{array} \right\} \pmod{3^{k+1}}.$$

Since $\frac{\pi(3^k)}{3} \equiv 0 \pmod{24}$, we have the lemma. □

Note that it is easy to see that if $F_n \equiv b \pmod{3^k}$, then there is a number m, $1 \leq m \leq 72$, such that $n \equiv m \pmod{72}$ and $F_m \equiv b \pmod{27}$.

Theorem 4.6: *For $k \geq 3$, $\nu(3^k, b) = 8$ if $b \equiv 1$ or $26 \pmod{27}$.*

Proof: We shall prove the theorem by induction on k. Consider $b \equiv 1 \pmod{27}$ first. Suppose $k = 3$. Then by Table 1 we have $\nu(3^3, 1) = 8$.

Suppose $\nu(3^k, b) = 8$ for $k \geq 3$. Let $b \in \mathbb{Z}_{3^{k+1}}$ with $b \equiv 1 \pmod{27}$. Let $F_{n_i} \equiv b \pmod{3^k}$, $1 \leq i \leq 8$ and $1 \leq n_i \leq \pi(3^k)$. Since $F_{n_i} \equiv 1 \pmod{27}$, it is easy to see from Table 1 that $n_i \not\equiv 6, 18 \pmod{24}$. By Lemmas 4.4 and 4.5 there are at least $\nu(3^k, b) = 8$ n_i''s with $0 \leq n_i' \leq \pi(3^{k+1})$ such that $F_{n_i'} \equiv b \pmod{3^{k+1}}$. Since there are 3^{k-2} solutions in $\mathbb{Z}_{3^{k+1}}$ for the congruence equation $b \equiv 1 \pmod{27}$, $\omega(3^{k+1}, 1) \geq 8 \cdot 3^{k-2}$. But it is known from Lemma 4.3 that $\omega(3^{k+1}, 1) = 8 \cdot 3^{k-2}$. Therefore $\nu(3^{k+1}, b) = 8$.

The proof for $b \equiv 26 \pmod{27}$ is similar. $\square$

By a similar proof we obtain the following theorem.

Theorem 4.7: *For $k \geq 3$, $\nu(3^k, b) = 2$ if $b \not\equiv 1, 8, 19$ nor $26 \pmod{27}$.*

It is easy to see that $\nu(3, 0) = 2$, $\nu(3, 1) = \nu(3, 2) = 3$ and $\nu(9, 1) = \nu(9, 8) = 5$ and $\nu(9, b) = 2$ for $b \neq 1$ nor 8.

In general we do not have a formula to describe the number $\nu(3^k, b)$ for $b \equiv 8, 19 \pmod{27}$ yet. Suppose $b = 27m + 8$ with $0 \leq m < 3^{k-3}$ and $b' \equiv -b \pmod{3^k}$. Then it is easy to see that $b' = 27m' + 19$ for some m'. Namely, $m' = 3^{k-3} - m - 1$. By Corollary 3.4, we have $\nu(3^k, 27m + 8) = \nu(3^k, 27m' + 19)$. Thus, we shall be only interested in $\nu(3^k, 27m + 8)$. We give below some numerical data for $\nu(3^k, 27m + 8)$ when $3 \leq k \leq 10$.

$\nu(3^3, 8) = 5$.

$\nu(3^4, 8) = 11$, $\nu(3^4, b) = 2$ otherwise.

$\nu(3^5, 8) = 20$, $\nu(3^5, 89) = 11$, $\nu(3^5, b) = 2$ otherwise.

	$\nu(3^6, 27m + 8)$
$m \equiv 0 \pmod{3^2}$	20
$m = 12$	29
otherwise	2

	$\nu(3^7, 27m + 8)$
$m \equiv 0 \pmod{3^2}$	20
$m = 12$	29
$m = 66$	56
otherwise	2

	$\nu(3^8, 27m + 8)$
$m \equiv 0 \pmod{3^2}$	20
$m \equiv 66 \pmod{3^4}$	56
$m = 12$	83
otherwise	2

	$\nu(3^9, 27m + 8)$
$m \equiv 0 \pmod{3^2}$	20
$m \equiv 66 \pmod{3^4}$	56
$m = 12$	83
$m = 498$	164
otherwise	2

	$\nu(3^{10}, 27m+8)$
$m \equiv 0 \pmod{3^2}$	20
$m \equiv 66 \pmod{3^4}$	56
$m \equiv 498 \pmod{3^6}$	164
$m = 741$	245
otherwise	2

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