PRODUCTS OF ELLIPTICAL CHORD LENGTHS AND THE FIBONACCI NUMBERS

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1. INTRODUCTION

In a recent article [4] we related the product of chord lengths of an ellipse to a collection of polynomials. Specifically, for $a \geq |b|$, the locus of points

$$ae^{i\theta} + be^{-i\theta} = (a+b)\cos\theta + i(a-b)\sin\theta, \quad 0 \leq \theta < 2\pi \tag{1}$$

where $i = \sqrt{-1}$ describes an ellipse with vertices $\pm(a+b)$ and $\pm i(a-b)$. The major axis lies on the y-axis if $b < 0$ (see figure 1) and on the x-axis if $b > 0$. The curve is a circle of radius a whenever $b = 0$. (In [4] we restricted $b \geq 0$.) Set $\theta_{n,j} := 2j\pi/n$ and $z_{n,j}^{(a,b)} := ae^{i\theta_{n,j}} + be^{-i\theta_{n,j}}$ for $j = 0, \ldots, n-1$.

Figure 1: An ellipse with $b < 0$

Note that the nodes $z_{n,j}^{(a,b)}$ are the image of the n^{th} roots of unity under the mapping

$$e^{i\theta} \mapsto ae^{i\theta} + be^{-i\theta}.$$

Define the polynomials

$$P_n(z; a, b) := \prod_{j=0}^{n-1}(z - z_{n,j}^{(a,b)}) + (a^n + b^n) \tag{2}$$

for all positive integers n. For convenience we set $P_0(z; a, b) = 2$. By direct substitution we observe that $P_n(z; a, b)$ is the unique monic polynomial interpolant to $a^n + b^n$ in the nodes $z_{n,j}^{(a,b)}$, $j = 0, \ldots, n-1$. Slight modifications of arguments given in [4] verify that the polynomials $P_n(z; a, b)$ satisfy the three-term recurrence relation

$$P_n(z; a, b) = zP_{n-1}(z; a, b) - abP_{n-2}(z; a, b), \quad n = 2, 3, \ldots., \tag{3}$$

and the identity

$$P_n(ae^{i\theta}+be^{-i\theta};a,b)=a^ne^{in\theta}+b^ne^{-in\theta},\quad 0\le\theta\le 2\pi\quad (n\ge 0). \tag{4}$$

The first few polynomials in this family are

$$\begin{aligned}
P_0(z;a,b)&=2\\
P_1(z;a,b)&=z\\
P_2(z;a,b)&=z^2-2ab\\
P_3(z;a,b)&=z^3-3abz\\
P_4(z;a,b)&=z^4-4abz^2+2a^2b^2\\
P_5(z;a,b)&=z^5-5abz^3+5a^2b^2z.
\end{aligned}$$

For convenience, set $z_0:=z_{n,0}^{(a,b)}=a+b$. Observe that in view of (2)

$$P_n(z_0;a,b)=a^n+b^n. \tag{5}$$

Hence,

$$P_n(z;a,b)-P_n(z_0;a,b)=\prod_{j=0}^{n-1}(z-z_{n,j}^{(a,b)}).$$

This suggests that the product of the lengths of the chords of the ellipse from z_0 to the points $z_{n,j}^{(a,b)}$, $j=1,\dots,n-1$ is given by the positive real number

$$\begin{aligned}
P_n'(z_0;a,b)&=\lim_{z\to z_0}\frac{P_n(z;a,b)-P_n(z_0;a,b)}{z-z_0} &(6)\\
&=\prod_{j=1}^{n-1}\left(z_0-z_{n,j}\right). &(7)
\end{aligned}$$

(See figure 2 for the case $n=8$.) In [4] we also established

$$P_n'(z_0;a,b)=n\frac{a^n-b^n}{a-b} \tag{8}$$

which, except for the factor of n, reminds us of Binet's formula for the Fibonacci numbers just as equation (5) is reminiscent of Binet's formula for the Lucas numbers. A recent and exhaustive treatise on these numbers can be found in [3].

Figure 2: Chords determined by $n=8$ points

We can now state our main result. Set

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} \tag{9}$$

so that $\alpha+\beta = -\alpha\beta = 1$. Note that α is the *golden ratio*. Then

$$F_n = \frac{1}{n} P_n'\left(1;\alpha,\beta\right), \quad n \in \mathbb{N} \tag{10}$$

where F_n denotes the n^{th} Fibonacci number. This result follows immediately from Binet's formula and equation (8). Likewise,

$$L_n = P_n\left(z_{n,j}^{(\alpha,\beta)};\alpha,\beta\right) = \alpha^n + \beta^n \tag{11}$$

for any $j = 0,\ldots,n-1$ where L_n represents the n^{th} Lucas number. In what follows we will derive equations (10) and (11) without the use of Binet's formula. We will also verify that the sequences $\{P_n\left(z;\alpha,\beta\right)\}$ and $\left\{\frac{1}{n}P_n'\left(z;\alpha,\beta\right)\right\}$ are the classical Lucas and Fibonacci polynomials respectively.

The goal of this note is to relate both the Lucas and Fibonacci sequences to products of chord lengths of an ellipse. Actually, we relate classes of generalized Fibonacci and Lucas sequences to elliptical chord lengths. These results will be used to establish Binet's formulas and divisibility properties of Fibonacci type sequences such as the well known identity

$$F_{2n} = F_n L_n.$$

We plan to elaborate further on these and other topics in subsequent articles.

2. GENERALIZED FIBONACCI AND LUCAS NUMBERS

The polynomials $P_n\left(z;a,b\right)$ and $\frac{1}{n}P_n'\left(z;a,b\right)$ will be referred to as the **generalized Lucas and Fibonacci** polynomials respectively. (That these definitions are not the most general possible will not concern us here.) These polynomials were examined in [1] and [2]. Interestingly, the generalized Fibonacci polynomials also satisfy the three term recurrence relation (3). Specifically,

$$\frac{1}{n}P_n'\left(z;a,b\right) = \frac{z}{n-1}P_{n-1}'\left(z;a,b\right) - \frac{ab}{n-2}P_{n-2}'\left(z;a,b\right). \tag{12}$$

Since we are dealing with polynomials it is sufficient to verify equation (12) for $z = ae^{i\theta} + be^{-i\theta}$, that is for z on the ellipse (1). Using the chain rule we have

$$\frac{dP_n}{d\theta} = \frac{dP_n}{dz}\frac{dz}{d\theta}$$

$$\Longrightarrow \frac{dP_n}{dz} = \frac{dP_n/d\theta}{dz/d\theta}$$

where by (4)

$$\frac{dP_n}{d\theta} = in\left(a^n e^{in\theta} - b^n e^{-in\theta}\right). \tag{13}$$

Hence, for $n > 2$

$$\begin{aligned}
\frac{z}{n-1}\frac{dP_{n-1}}{dz} - \frac{ab}{n-2}\frac{dP_{n-2}}{dz} &= \frac{z}{n-1}\frac{dP_{n-1}/d\theta}{dz/d\theta} - \frac{ab}{n-2}\frac{dP_{n-2}/d\theta}{dz/d\theta} \\
&= \left(\frac{z}{n-1}\frac{dP_{n-1}}{d\theta} - \frac{ab}{n-2}\frac{dP_{n-2}}{d\theta}\right)/\frac{dz}{d\theta} \\
&= i\left(a^n e^{i\theta n} - b^n e^{-i\theta n}\right)/\frac{dz}{d\theta} \\
&= \frac{1}{n}\frac{dP_n/d\theta}{dz/d\theta} \\
&= \frac{1}{n}\frac{dP_n}{dz}.
\end{aligned}$$

Equation (12) follows.

Definition 1: *Two real numbers a and b with $a \geq |b|$ are* **suitable** *if both $a+b$ and ab are integers.*

Observe that a and b are suitable if and only if there are integers u and v such that

$$a = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2 - 4v}$$

$$b = \frac{1}{2}u - \frac{1}{2}\sqrt{u^2 - 4v}$$

with $u^2 \geq 4v$.

Suppose a and b are suitable. Then $P_1(z_0) = a+b$ and $P_2(z_0) = (a+b)^2 - 2ab = a^2 + b^2$ are integers. These values coupled with those obtained using the recurrence relation (3) generate the Fibonacci-like *integer* sequence

$$a+b, a^2+b^2, a^3+b^3, \dots .$$

We designate the n^{th} term of this sequence by $L_n^{(a,b)}$. It is easy to verify that if a and b are given by (9) then the above sequence reduces to the classical Lucas sequence $\{L_n\}$. Similarly, since $P_1'(z_0) = 1$ and $\frac{1}{2}P_2'(z_0) = a+b$, equation (12) defines an integer sequence $\left\{F_n^{(a,b)}\right\}$ whenever a and b are suitable. If a and b are given by (9) then $\left\{F_n^{(a,b)}\right\}$ reduces to the classical Fibonacci sequence $\{F_n\}$.

Recall that $P_1(z;\alpha,\beta) = z$ and $P_2(z;\alpha,\beta) = z^2 - 1$ and note that these are the first two classical Lucas polynomials. Also, P_1' and P_2' are the first two classical Fibonacci polynomials. Since $P_n(z;\alpha,\beta)$ and $\frac{1}{n}P_n'(z;\alpha,\beta)$ satisfy the three-term recurrence relation used to generate the Fibonacci and Lucas polynomials (see (3) and (12) respectively), it must follow that $P_n(z;\alpha,\beta)$ and $\frac{1}{n}P_n'(z;\alpha,\beta)$ are the classical Fibonacci and Lucas polynomials.

Binet's formulas for these generalized polynomials are easily established. Indeed, equation (5) gives the generalized Binet's formula for the sequence $\left\{L_n^{(a,b)}\right\}$. Using equation (13) we have

$$\left.\frac{dP_n}{dz}\right|_{z=z_0} = \left.\frac{dP_n/d\theta}{dz/d\theta}\right|_{\theta=0}$$

$$= \left.\frac{in\left(a^n e^{in\theta} - b^n e^{-in\theta}\right)}{i\left(ae^{i\theta} - be^{-i\theta}\right)}\right|_{\theta=0}$$

from which the Binet's formula for the generalized Fibonacci sequence follows.

The surprising thing is that by equation (7) $nF_n^{(a,b)}$ represents the product of lengths of chords of an ellipse determined by the base point z_0 and the the points $z_{n,j}^{(a,b)}, j = 1, \ldots, n-1$. This observation provides us with method to geometrically substantiate some of the divisibility properties of the generalized Fibonacci numbers. The following example illustrates this approach.

Example 2: *It is known that $F_3 \mid F_{3n}$ for any positive integer n ([3]). A similar result holds for the generalized Fibonacci numbers. Since the two chords used to derive $F_3^{(a,b)}$ also appear in the product of $F_{3n}^{(a,b)}$ (because the 3^{rd} roots of unity are also $3n^{th}$ roots of unity), $\frac{1}{3n}P_{3n}'(z_0;a,b) = \frac{1}{3}P_3'(z_0;a,b)\cdot l$ where l must be in integer. Hence, $\frac{1}{3}P_3'(z_0;a,b) = F_3^{(a,b)}$ will always be a factor of $\frac{1}{3n}P_{3n}'(z_0;a,b) = F_{3n}^{(a,b)}$.*

The method used in the last example provides us with an approach for proving a more general result.

Proposition 3: *Let m and n be positive integers. If m divides n then $F_m^{(a,b)}$ divides $F_n^{(a,b)}$.*

Proof: *Suppose m divides n. Then the m^{th} roots of unity are also n^{th} roots of unity. This ensures the containment*

$$\left\{z_{m,j}^{(a,b)}\right\} \subset \left\{z_{n,j}^{(a,b)}\right\}$$

which means that the chords appearing in the product representing $\frac{1}{m}P_m'(z_0;a,b)$ also appear in the representation of $\frac{1}{n}P_n'(z_0;a,b)$. Consequently,

$$\frac{1}{n}P_n'(z_0;a,b) = \frac{1}{m}P_m'(z_0;a,b)\cdot l$$

where l must be an integer. The desired result follows. □

Corollary 4: *If m and n are positive integers for which $\gcd(m,n) = 1$, then $F_m^{(a,b)}F_n^{(a,b)}$ divides $F_{mn}^{(a,b)}$.*

Proof: *Except for* 1 *no* m^{th} *root of unity is an* n^{th} *root of unity or vice-versa whenever* $\gcd(m,n) = 1$. *Hence, all chord lengths in the product forming* $\frac{1}{m}P'_m(z_0;a,b)\,\frac{1}{n}P'_n(z_0;a,b)$ *must also appear in the product* $\frac{1}{mn}P'_{mn}(z_0;a,b)$. $\square$

3. GENERALIZED LUCAS NUMBERS AS PRODUCTS OF CHORD LENGTHS

The generalized Lucas numbers are also expressible as elliptical chord lengths. To see this we first fix $\psi \in [0, 2\pi)$ and rotate the endpoints of line segments on the ellipse using the transformation that maps

$$e^{i\theta} \longrightarrow ae^{i(\theta+\psi)} + be^{-i(\theta+\psi)}.$$

Observe that this transformation does *not* rotate each point on the ellipse by the angle ψ. Set

$$z^{(a,b)}_{n,j,\psi} := ae^{i(\theta_{n,j}+\psi)} + be^{-i(\theta_{n,j}+\psi)},\ j = 0, \dots, n-1.$$

Because $z^{(a,b)}_{n,j,0} = z^{(a,b)}_{n,j}$ we will suppress the additional subscript whenever $\psi = 0$. In view of Equation (4),

$$P_n(z^{(a,b)}_{n,j,\psi}; a, b) = a^n e^{in\psi} + b^n e^{-in\psi},\ j = 0, 1, \dots, n-1.$$

Consequently, $P_n(z;a,b)$ is the monic polynomial interpolant to $a^n e^{in\psi} + b^n e^{-in\psi}$ in the nodes $z^{(a,b)}_{n,j,\psi}$, $j = 0, \dots, n-1$. Hence,

$$P_n(z;a,b) - \left(a^n e^{in\psi} + b^n e^{-in\psi}\right) = \prod_{j=0}^{n-1}\left(z - z^{(a,b)}_{n,j,\psi}\right) \tag{14}$$

has roots $z^{(a,b)}_{n,j,\psi}$, $j = 0, 1, \dots, n-1$. Next, for a positive integer n (using $\psi = \pi/n$) set

$$Q_n(z;a,b) = P^{(a,b)}_n(z;a,b) + a^n + b^n \tag{15}$$

$$= \prod_{j=0}^{n-1}\left(z - z^{(a,b)}_{n,j,\pi/n}\right). \tag{16}$$

There are two facts worth noting about these polynomials. First, in view of Equations (5 and 15)

$$Q_n(z_0;a,b) = P_n(a+b;a,b) + a^n + b^n = 2L^{(a,b)}_n.$$

This means that $2L^{(a,b)}_n$ is the product of the lengths of the elliptical chords determined by the base point z_0 and the points $z^{(a,b)}_{n,j,\pi/n}, j = 0, 1, \dots, n-1$ since by equation (16) $z - z^{(a,b)}_{n,j,\pi/n}$, $j = 0, 1, \dots, n-1$ are the factors of $Q_n(z;a,b)$. Second, the roots $z^{(a,b)}_{n,j,\pi/n}, j = 0, 1, \dots, n-1$ of $Q_n(z;a,b)$ satisfy

$$z^{(a,b)}_{n,j,\pi/n} = z^{(a,b)}_{2n,2j+1}, j = 0, 1, \dots, n-1.$$

(See figure 3 for the case $n = 4$.) Consequently, the chords appearing in the product $P'_{2n}(z_0; a, b)$ are precisely those used in the products determining $P'_n(z_0; a, b)$ and $Q_n(z_0; a, b)$. This establishes the well-known identity

$$\begin{aligned} F_{2n}^{(a,b)} &= \frac{1}{2n} P'_{2n}(z_0; a, b) \\ &= \left(\frac{1}{n} P'_n(z_0; a, b)\right)\left(\frac{1}{2} Q_n(z_0; a, b)\right) \\ &= F_n^{(a,b)} L_n^{(a,b)}. \end{aligned}$$

Figure 3: Relationship between $z_{n,j,\pi/n}^{(a,b)}$ and $z_{2n,2j+1}^{(a,b)}$ $(n = 4)$

Several divisibility properties for generalized Lucas numbers can be established using the same strategy employed for the generalized Fibonacci numbers above. For example, since the n^{th} roots of (-1) are also $\left(3n^{th}\right)$ roots of (-1), we easily can establish that $L_n^{(a,b)}$ divides $L_{3n}^{(a,b)}$ for all positive integers n. Some readers may enjoy using these elliptical chord length representations to verify additional properties of the generalized Fibonacci and Lucas numbers. The author of this note is currently conducting a seminar on the Fibonacci numbers with an aim toward student research using this approach. In addition, he is interested in the properties of the generalized polynomials.

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