PRODUCT DIFFERENCE FIBONACCI IDENTITIES OF SIMSON, GELIN-CESARO, TAGIURI AND GENERALIZATIONS

Steve Fairgrieve
Department of Mathematics, West Virginia University, PO Box 6310, Morgantown, WV 26506-6310

Henry W. Gould
Department of Mathematics, West Virginia University, PO Box 6310, Morgantown, WV 26506-6310
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Dedicated to Professor Sam B. Nadler, Jr. on his 65th birthday.

By a Product Difference Fibonacci Identity (PDFI) we mean something of the form

$$\prod_{i=1}^{s} F_{n+a_i} - \prod_{i=1}^{s} F_{n+b_i} = D_n(a_i, b_i; s) = D_n \tag{1}$$

where $s \geq 1$, a_i and b_i are specified integers and D_n is of some interesting form for all integers n.

Here we shall be concerned primarily with the case where all the b's are zero so that the second product is just F_n^s. The most well-known example is the formula

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{2}$$

which Dickson [2, p. 393] credits to Robert Simson in 1753. This may be generalized to

$$F_{n-r}F_{n+r} - F_n^2 = (-1)^{n+r+1}F_r^2 \tag{3}$$

which is essentially the form found by Catalan [1].

Still more general is the beautiful formula

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_a F_b \tag{4}$$

which Dickson [2, p. 404] credits to A. Tagiuri in 1901.

We should remark that Gould [3] exhibited an extension of Tagiuri's identity to products of three Fibonacci numbers in the form

$$\begin{aligned} F_{n+a}F_{n+b}F_{n+c} - F_nF_{n+a}F_{n+b+c} + F_nF_{n+b}F_{n+c+a} - F_nF_{n+c}F_{n+a+b} \\ = (-1)^n \left\{F_aF_bF_{n+c} - F_cF_aF_{n+b} + F_bF_cF_{n+a}\right\}. \end{aligned} \tag{5}$$

If we designate this by $(-1)^n K_n$, then we see that K_n is recursive and satisfies the Fibonacci recursion $K_{n+1} = K_n + K_{n-1}$. A complicated formula was given in [3] for K_n that is related to the standard Binet formula.

A search of the literature turns up very little about product difference Fibonacci identities of the form (1) for $s \geq 3$. Recently, however, Melham [4] has discovered the attractive formula

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n. \tag{6}$$

We have run extensive computer searches and found the attractive companion formula

$$F_n F_{n+4} F_{n+5} - F_{n+3}^3 = (-1)^{n+1} F_{n+6}. \tag{7}$$

To bring out a duality between these, we prefer to rewrite (6) in the form

$$F_{n-2} F_{n-1} F_{n+3} - F_n^3 = (-1)^{n-1} F_{n-3}. \tag{8}$$

and rewrite (7) in the form

$$F_{n+2} F_{n+1} F_{n-3} - F_n^3 = (-1)^n F_{n+3}. \tag{9}$$

Then, recalling the relation between Fibonacci numbers with positive subscripts and those with negative subscripts

$$F_{-n} = (-1)^{n+1} F_n, \tag{10}$$

we see by replacing n with $-n$ then (8) yields (9) and conversely, so that they may be considered as duals. The proof of one implies the other.

It is easy to see by using (10) that relations (2) and (3) are each self-duals.

Our computer search turned up another pair of cubic dual identities which are readily proved by induction:

$$F_{n-2} F_{n+1}^2 - F_n^3 = (-1)^{n-1} F_{n-1} \tag{11}$$

and

$$F_{n+2} F_{n-1}^2 - F_n^3 = (-1)^n F_{n+1}. \tag{12}$$

Our computer search turned up no other non-trivial examples where

$$F_{n+a} F_{n+b} F_{n+c} - F_n^3 \tag{13}$$

has any simple form.

Looking back at (1), when $s = 4$, there is the very attractive formula

$$F_{n-2} F_{n-1} F_{n+1} F_{n+2} - F_n^4 = -1, \tag{14}$$

which Dickson [2, p. 401] reports was stated by E. Gelin and proved by Cesàro. Just as with (2) and (3) we see by using (10) that (14) is self-dual.

Thinking back to relations (11) and (12) we looked for something of that sort for fourth powers and we are able to announce

$$\begin{aligned} F_{n-3} F_{n+1}^3 - F_n^4 &= (-1)^n \{F_n F_{n+3} + F_{n-3} F_{n+1}\} \\ &= (-1)^n \left\{F_{n-1} F_{n+3} + 2F_n^2\right\} \end{aligned} \tag{15}$$

and its natural dual

$$\begin{aligned} F_{n+3} F_{n-1}^3 - F_n^4 &= (-1)^{n-1} \{F_n F_{n-3} - F_{n+3} F_{n-1}\} \\ &= (-1)^n \left\{F_n^2 + F_n F_{n-1} + 2F_{n-1}^2\right\}. \end{aligned} \tag{16}$$

There are other forms for the right members of these but none seems to reduce to a single term. Relations (15) and (16) are readily proved by induction, and of course the one implies the other.

It is natural to move up to the case $s = 5$. Nothing quite as nice seems to exist. Here is one example which is easily proved:

$$F_{n-1}^3 F_{n+1} F_{n+2} - F_n^5 = -F_n + (-1)^n F_{n-1} F_{n+1} F_{n+2}. \tag{17}$$

One way to obtain a sixth power identity is to use the pair of dual third power identities (8) and (9) and relation (3) with $r = 3$ as follows:

$$\begin{aligned} F_{n-3}F_{n-2}F_{n-1}F_{n+1}F_{n+2}F_{n+3} &= \left(F_{n-2}F_{n-1}F_{n+3}\right)\left(F_{n+2}F_{n+1}F_{n-3}\right) \\ &= \left(F_n^3 - (-1)^n F_{n-3}\right)\left(F_n^3 + (-1)^n F_{n+3}\right) \\ &= F_n^6 - (-1)^n F_{n-3}F_n^3 + (-1)^n F_{n+3}F_n^3 - F_{n-3}F_{n+3} \\ &= F_n^6 - (-1)^n F_{n-3}F_n^3 + (-1)^n F_{n+3}F_n^3 - \left(F_n^2 + 4(-1)^n\right) \\ &= F_n^6 - (-1)^n F_{n-3}F_n^3 + (-1)^n F_{n+3}F_n^3 - F_n^2 - 4(-1)^n, \end{aligned}$$

which gives us the identity

$$F_{n-3}F_{n-2}F_{n-1}F_{n+1}F_{n+2}F_{n+3} - F_n^6 = (-1)^n F_n^3 \left(F_{n+3} - F_{n-3}\right) - F_n^2 - 4(-1)^n, \tag{18}$$

which may be simplified further to give

$$F_{n-3}F_{n-2}F_{n-1}F_{n+1}F_{n+2}F_{n+3} - F_n^6 = (-1)^n \left\{4F_n^4 - (-1)^n F_n^2 - 4\right\}. \tag{19}$$

It is easy to see from this that the PDFI is positive for even $n \geq 4$ and negative for odd $n \geq 1$. For $n = 0$ it is -4 and for $n = 2$ it is -1.

A shorter derivation of (19) is as follows:

$$\left(F_{n-3}F_{n+3}\right)\left(F_{n-2}F_{n+2}F_{n-1}F_{n+1}\right) = \left(F_n^2 + 4(-1)^n\right)\left(F_n^4 - 1\right) = F_n^6 - F_n^2 + 4(-1)^n F_n^4 - 4(-1)^n,$$

which gives (19) at once.

We now introduce the notation

$$P_r(n) = \left(F_{n-r}F_{n+r}\right) P_{r-1}(n), \quad r \geq 2 \tag{20}$$

with

$$P_1(n) = F_{n-1}F_{n+1}.$$

Then using

$$F_{n-r}F_{n+r} = F_n^2 + (-1)^{n+r+1} F_r^2, \tag{21}$$

which is a rephrasing of (3), we have a way to generate as many formulas as we like. Examples:

$$P_4(n) = F_n^8 - 5(-1)^n F_n^6 - 37F_n^4 + 5(-1)^n F_n^2 + 36, \tag{22}$$

and

$$P_5(n) = F_n^{10} + 20(-1)^n F_n^8 - 162F_n^6 - 920(-1)^n F_n^4 + 161F_n^2 + 900(-1)^n. \tag{23}$$

Continuing in this manner one may generate higher power PDFI's as far as desired. We now give a general result.

Theorem: Using the notation (20), there exist integer coefficients Q_i^r such that

$$P_r(n) = \sum_{i=0}^{r} (-1)^{n(r-i)} Q_i^r F_n^{2i} \tag{24}$$

where the Q's satisfy the recurrence relation

$$Q_i^{r+1} = Q_{i-1}^r + (-1)^r F_{r+1}^2 Q_i^r, \tag{25}$$

where $Q_i^r = 0$ if $i < 0$ or $i > r$, and $Q_r^r = 1$.

The proof is by simple mathematical induction.

Using the techniques in [4] it would also be a straightforward exercise to extend the identities we have presented to hold in suitable form for the generalized Fibonacci sequence $\{W_n\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b \tag{26}$$

but we will not take the space to do this here.

If we take all the b's to be zero in (1) then we raise the natural question from what we have presented so far as to when the expression

$$D_n = D_n(a_i, 0, s) = \prod_{i=1}^{s} F_{n+a_i} - F_n^s \tag{27}$$

has a simple and interesting form. Since we seek a nice form for all integers n, it would be sufficient to consider the case when $n = 0$, in which case $F_0^s = 0$, and examine how $\prod_{i=1}^{s} F_{a_i}$ may be rewritten.

It is known, for example, that the product of just two Fibonacci numbers may not be expressed in general as a Fibonacci number itself. But as (15), (16), (17) and (18) show, it is not to be expected that a PDFI will have the form of a simple product. The non-symmetrical PDFI's for $s = 3$ and 5, and the symmetrical forms for $s = 2, 4, 6$ suggest that only when s is even may we expect in general to find nice symmetrical self-dual PDFI's. For odd powers each PDFI will clearly be paired with a dual. Each Factor F_{n-a} will be paired in the dual with the factor F_{n+a} just as in the pairs (8) and (9).

Remark: Relation (5), while not very symmetrical in itself may be used by permuting the parameters a, b, and c to obtain the symmetrical formula

$$\begin{aligned} 3F_{n+a}F_{n+b}F_{n+c} - F_nF_{n+a}F_{n+b+c} - F_nF_{n+b}F_{n+c+a} - F_nF_{n+c}F_{n+a+b} \\ = (-1)^n \{F_aF_bF_{n+c} + F_bF_cF_{n+a} + F_cF_aF_{n+b}\}. \end{aligned} \tag{28}$$

Similar complicated formulas exist for products of s Fibonacci numbers.

	0	1	2	3	4	5	6	7 ⋯ i
1	1	1						
2	1	0	−1					
3	1	4	−1	−4				
4	1	−5	−37	5	36			
5	1	20	−162	−920	161	900		
6	1	−44	−1442	9448	59041	−9404	−57600	
7	1	125	−8878	−234250	1655753	9968525	−1646876	−9734400
r								

Table 1. Values of Q^r_{r-i} for $1 \leq r \leq 7, \quad 0 \leq i \leq r$.

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