THE LUCAS TRIANGLE REVISITED

Neville Robbins
Mathematics Department, San Francisco State University, San Francisco, CA 94132

(Submitted August 2002-Final Revision December 2002)

1. INTRODUCTION

The Lucas triangle is an infinite triangular array of natural numbers that is a variant of Pascal's triangle. In this note, we prove a property of the Lucas triangle that has been merely stated by prior researchers; we also present some apparently new properties of the Lucas triangle.

2. PASCAL'S TRIANGLE

We begin by reviewing some properties of the triangular array of natural numbers known as Pascal's triangle. The n^{th} row of Pascal's triangle consists of entries denoted $\binom{n}{k}$ where n and k are integers such that $n \geq 1$ and $0 \leq k \leq n$. The first 8 rows of Pascal's triangle are presented below in left-justified format:

1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

The entries $\binom{n}{k}$ (usually called "n choose k") are known as *binomial coefficients*. The following properties of binomial coefficients are well-known:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1}$$

$$\binom{n}{n-k} = \binom{n}{k} \qquad (symmetry) \tag{2}$$

$$\binom{n}{0} = \binom{n}{n} = 1 \tag{3}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \tag{5}$$

$$\sum_{k\,odd} \binom{n}{k} = \sum_{k\,even} \binom{n}{k} = 2^{n-1}. \tag{6}$$

$$If \quad 1 \leq k \leq n, \quad then \tag{7}$$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \qquad (Pascal's \quad identity)$$

$$p \quad is \quad prime \quad if \quad and \quad only \quad if \tag{8}$$

$$p| \binom{p}{k} \quad \forall k \quad such \quad that \quad 1 \leq k \leq p-1.$$

In addition, there are identities that link binomial coefficients to Fibonacci numbers, namely:

$$F_{n+1} = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i}; \tag{9}$$

$$F_{2n} = \sum_{i=0}^{n-1} \binom{n+i}{1+2i}; \tag{10}$$

$$F_{2n+1} = \sum_{i=0}^{n} \binom{n+i}{2i}. \tag{11}$$

Note that Pascal's triangle could be generated inductively using only (3) and (7).

The following definitions are useful in determining the highest power of a given prime that divides a binomial coefficient.

Definition 1: If p is prime and the integer $m \geq 2$, let $o_p(m) = k \geq 0$ if k is the unique integer such that $p^k | m$, $p^{k+1} \not| m$.

Proposition 1: If a and $b \in N$, then $o_p(ab) = o_p(a) + o_p(b)$.

Proposition 2: If a, b, and $a/b \in N$, then $o_p(a/b) = o_p(a) - o_p(b)$.

Definition 2: If p is a prime and the $m \geq 2$, let the representation of m to the base p be given by:

$$m = \sum_{i=0}^{r} a_i p^i \quad where \quad 0 \leq a_i \leq p-1 \quad \forall i \,,\, a_r \neq 0.$$

Definition 3: With p and m as in Definition 2, let $t_p(m)$ denote the sum of the digits of m to the base p, that is

$$t_p(m) = \sum_{i=0}^{r} a_i.$$

Proposition 3: If p is prime and $0 \leq k \leq n$, then

$$o_p\left(\binom{n}{k}\right) = \frac{t_p(k) + t_p(n-k) - t_p(n)}{p-1}.$$

Proposition 4: If $m \geq 1$, then $\binom{2^m}{k}$ is even for all k such that $1 \leq k \leq 2^m - 1$.

Proposition 5: If $m \geq 2$, then $\binom{2^m - 1}{k}$ is odd for all k such that $1 \leq k \leq 2^m - 2$.

Remarks: Propositions 1 and 2 follow easily from Definition 1. Proposition 3 follows from [4] Theorem 3.15 , p. 54. (See also [4], exercises 31-32, p. 55-56.) Propositions 4 and 5 follow from Proposition 3.

3. THE LUCAS TRIANGLE

The Lucas triangle is an infinite triangular array of natural numbers whose n^{th} row consists of entries that we will denote $\begin{bmatrix} n \\ k \end{bmatrix}$, where $n \geq 1$ and $0 \leq k \leq n$. The symbol $\begin{bmatrix} n \\ k \end{bmatrix}$ may be defined inductively as follows:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1; \qquad \begin{bmatrix} n \\ n \end{bmatrix} = 2. \tag{12}$$

$$If \quad 1 \leq k \leq n, \quad then \tag{13}$$

$$\begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k \end{bmatrix}.$$

The first eight rows of the Lucas triangle are presented below, in left-justified format:

1	2							
1	3	2						
1	4	5	2					
1	5	9	7	2				
1	6	14	16	9	2			
1	7	20	30	25	11	2		
1	8	27	50	55	36	13	2	
1	9	35	77	105	91	49	15	2.

Note that whereas Pascal's triangle is generated by the coefficients of $(a+b)^n$, the Lucas triangle is generated by the coefficients in the expansion of $(a+b)^{n-1}(a+2b)$.

Below, we list some properties of the Lucas triangle. Note that (13), (19), (20), (21), (22) are analogues of (7), (4), (5), (6), (8) respectively.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n+k}{n}\binom{n}{k} \tag{14}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k} + \binom{n-1}{k-1} \quad if \quad n \geq 2 \quad and \quad k \geq 1 \tag{15}$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = n + 1 \tag{16}$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = 2n - 1 \tag{17}$$

$$\begin{bmatrix} n \\ n-2 \end{bmatrix} = (n-1)^2 \quad if \quad n \geq 2 \tag{18}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = 3(2^{n-1}) \tag{19}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad if \quad n \geq 2. \tag{20}$$

$$\sum_{k\, even} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k\, odd} \begin{bmatrix} n \\ k \end{bmatrix} = 3(2^{n-2}) \quad if \quad n \geq 2. \tag{21}$$

$$p \quad is \quad prime \quad if \quad and \quad only \quad if \tag{22}$$

$$p \Big| \begin{bmatrix} p-i \\ i \end{bmatrix} \quad \forall i \quad such \quad that \quad 1 \leq i \leq \left[\frac{p}{2}\right].$$

Remarks: Identity (14) follows from (12), (13), (3), (7), and induction on n. Each of (15), (16), (17), (18) follow from (14); (19) follows from (15) and (4), while (20) follows from (15) and (5), as we shall demonstrate below. Identity (21) follows from (19) and (20), while (22) is Theorem 2 in [2]. We note that (15), (18), and (19) were stated without proof in [1].

In addition, the following identities link $\begin{bmatrix} n \\ k \end{bmatrix}$ to Fibonacci and Lucas numbers:

$$L_n = \sum_{i=0}^{[\frac{n}{2}]} \begin{bmatrix} n-i \\ i \end{bmatrix}; \tag{23}$$

$$F_{2n} = \sum_{i=0}^{n} \begin{bmatrix} n+i \\ 2i \end{bmatrix}; \tag{24}$$

$$F_{2n+1} = \sum_{i=0}^{n-1} \begin{bmatrix} n+i \\ 1+2i \end{bmatrix}. \tag{25}$$

Note that (23), (24), (25) are analogues of (9), (10), (11) respectively.

4. NEW RESULTS

We begin by proving identity (23), which has been previously hinted diagrammatically in [1] and stated without proof in [3]. Just as the sums of rising diagonals in Pascal's triangle yield the Fibonacci numbers, so do the rising sums of diagonals in the Lucas triangle yield the Lucas numbers.

Theorem 1: If $n \geq 1$, then

$$L_n = \sum_{i=0}^{[\frac{n}{2}]} \begin{bmatrix} n-i \\ i \end{bmatrix}.$$

Proof: (Induction on n) We note that $L_1 = 1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $L_2 = 3 = 1+2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the statement holds for $n = 1, 2$. Now

$$L_{n+2} = L_{n+1} + L_n = \sum_{i=0}^{[\frac{n+1}{2}]} \begin{bmatrix} n+1-i \\ i \end{bmatrix} \quad + \sum_{i=0}^{[\frac{n}{2}]} \begin{bmatrix} n-i \\ i \end{bmatrix}$$

by induction hypothesis. Therefore, we have

$$L_{n+2} = 1 + \sum_{i=1}^{[\frac{n+1}{2}]} \begin{bmatrix} n+1-i \\ i \end{bmatrix} + \sum_{i=1}^{[\frac{n}{2}]+1} \begin{bmatrix} n+1-i \\ i-1 \end{bmatrix}.$$

If $n = 2m - 1$, then we have

$$L_{2m+1} = 1 + \sum_{i=1}^{m} \begin{bmatrix} 2m-i \\ i \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 2m-i \\ i-1 \end{bmatrix}$$

$$= 1 + \sum_{i=1}^{m} \left(\begin{bmatrix} 2m-i \\ i \end{bmatrix} + \begin{bmatrix} 2m-i \\ i-1 \end{bmatrix} \right).$$

Now (13) implies

$$L_{2m+1} = 1 + \sum_{i=1}^{m} \begin{bmatrix} 2m+1-i \\ i \end{bmatrix} = \sum_{i=0}^{m} \begin{bmatrix} 2m+1-i \\ i \end{bmatrix}.$$

If $n = 2m$, then we have

$$L_{2m+2} = 1 + \sum_{i=1}^{m} \begin{bmatrix} 2m+1-i \\ i \end{bmatrix} + \sum_{i=1}^{m+1} \begin{bmatrix} 2m+1-i \\ i-1 \end{bmatrix}$$

$$= 1 + \sum_{i=1}^{m} \begin{bmatrix} 2m+1-i \\ i \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 2m+1-i \\ i-1 \end{bmatrix} + 2$$

$$= 1 + \sum_{i=1}^{m} \left(\begin{bmatrix} 2m+1-i \\ i \end{bmatrix} + \begin{bmatrix} 2m+1-i \\ i-1 \end{bmatrix} \right) + 2.$$

Now (13) implies

$$L_{2m+2} = 1 + \sum_{i=1}^{m} \begin{bmatrix} 2m+2-i \\ i \end{bmatrix} + 2 = \sum_{i=0}^{m+1} \begin{bmatrix} 2m+2-i \\ i \end{bmatrix}. \quad \square$$

The proofs of identities (24) and (25) are similar, and are therefore omitted. Next, we prove identity (20).

Theorem 2:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad if \quad n \geq 2$$

Proof: Invoking (18) and (5), we have

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 1 + \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 1 + \sum_{k=1}^{n} (-1)^k \left(\binom{n}{k} + \binom{n-1}{k-1} \right)$$

$$= 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} =$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+1} \binom{n-1}{j} = 0. \quad \square$$

The next theorem concerns "rising diagonals" in the Lucas triangle and is somewhat reminiscent of identity (22):

Theorem 3: If p is an odd prime, then

$$o_p\left(\begin{bmatrix} 2p-i \\ i \end{bmatrix} \right) = \begin{cases} 1 \text{ if } & 1 \leq i \leq \frac{p-1}{2} \\ 2 \text{ if } & \frac{p-1}{2} < i \leq p-1 \end{cases}.$$

Proof: Identities (17) and (1) imply

$$\begin{bmatrix} 2p-i \\ i \end{bmatrix} = \frac{2p}{i} \binom{2p-i}{i} = \frac{2p(2p-i)!}{i(i!)(2p-2i)!}.$$

Since $1 \leq i \leq p-1$ by hypothesis, it follows that $o_p(2p(2p-i)!) = 2$. Now

$$o_p(i(i!)(2p-2i)!) = o_p((2p-2i)!) = \begin{cases} 1 \text{ if } & 1 \leq i \leq \frac{p-1}{2} \\ 0 \text{ if } & \frac{p-1}{2} < i \leq p-1 \end{cases}.$$

The conclusion now follows from Proposition 2. □

The following theorem describes a row property enjoyed by odd primes.

Theorem 4: If p is an odd prime, then for all j such that $1 \le j \le p-1$ and for all i such that $j+1 \le i \le p-1$, we have $p \left| \begin{bmatrix} p+j \\ i \end{bmatrix} \right.$.

Proof: (Induction on j) Identities (17) and (1) imply

$$\begin{bmatrix} p+1 \\ i \end{bmatrix} = \frac{p+1+i}{i}\binom{p+1}{i} = \frac{(p+1+i)(p+1)!}{i(i!)(p+1-i)!}.$$

Since $2 \le i \le p-1$, it follows that p divides the numerator, but not the denominator of the latter fraction. Therefore the theorem holds for $j=1$. Now (13) implies that

$$\begin{bmatrix} p+j+1 \\ i \end{bmatrix} = \begin{bmatrix} p+j \\ i \end{bmatrix} + \begin{bmatrix} p+j \\ i-1 \end{bmatrix}.$$

By the induction hypothesis, each of the summands of the right member is divisible by p. Therefore the left member is divisible by p, so we are done. □

The final theorem concerns the parity of Lucas triangle entries in rows such that the row number is a power of 2.

Theorem 5: $\begin{bmatrix} 2^m \\ k \end{bmatrix}$ is odd for all k such that $1 \le k \le 2^m - 1$.

Proof: It suffices to invoke (18) with $n = 2^m$, and then make use of Propositions 4 and 5. □

REFERENCES

[1] Mark Feinberg. "A Lucas Triangle." *Fibonacci Quart.* **5** (1967): 486-490.

[2] H. W. Gould & W. E. Greig. "A Lucas Triangle Primality Test Dual to That of Mann-Shanks." *Fibonacci Quart.* **23** (1985): 66-69.

[3] Verner E. Hoggatt Jr. "An Application of the Lucas Triangle." *Fibonacci Quart.* **8** (1970): 360-364, 427.

[4] N. Robbins. *Beginning Number Theory.* (1993) Wm. C. Brown Publishers (Dubuque, IA).

AMS Classification Numbers: 11B65, 11B39

✠ ✠ ✠