ON THE K^{TH}–ORDER DERIVATIVE SEQUENCES OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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(Submitted January 2003-Final Revision July 2003)

ABSTRACT

In this note we consider two classes of polynomials U_n and V_n. These polynomials are special cases of $U_{n,m}$ and $V_{n,m}$ (see [2]), respectively. Also, U_n and V_n are generalized Fibonacci and Lucas polynomials. In fact, in this paper we study the polynomials $U_{n,3}$ and $V_{n,3}$, together with their k^{th}−derivative sequences $U_n^{(k)}$ and $V_n^{(k)}$. Some interesting identities are proved in the paper, for $U_n,\ V_n,\ U_n^{(k)}$ and $V_n^{(k)}$.

1. INTRODUCTION

To begin with, we define two classes of polynomials $\{U_n \equiv U_n(x)\}_{n\in N}$ and $\{V_n \equiv V_n(x)\}_{n\in N}$. These polynomials are given by recurrence relations:

$$U_n = xU_{n-1} + U_{n-m}, \quad n \geq m,$$

with $U_0 = 0, U_n = x^{n-1},\ n = 1, \ldots, m-1$, and

$$V_n = xV_{n-1} + V_{n-m}, \quad n \geq m,$$

with $V_0 = 2, V_n = x^n,\ n = 1, \ldots, m-1$.

These polynomials are special cases of the polynomials $U_{n,m}$ and $V_{n,m}$ (see [2], for $y = 1$). For $m = 2,\ U_n$ and V_n are the well-known Fibonacci and Lucas polynomials, respectively (see [3], [4], [5], [6], [7]).

In this paper we shall consider these polynomials for $m = 3$. Obviously, we can say that U_n and V_n are generalized Fibonacci and generalized Lucas polynomials. Namely, they are given by recurrence relations:

$$U_n = xU_{n-1} + U_{n-3}, \quad n \geq 3, \tag{1.1}$$

with $U_0 = 0, U_1 = 1, U_2 = x$, and

$$V_n = xV_{n-1} + V_{n-3}, \quad n \geq 3, \tag{1.2}$$

with $V_0 = 2, V_1 = x, V_2 = x^2$.

Recall that U_n is a special case of the polynomials $\phi_n(p, q; x)$ (see [1], for $p = 0, q = -1$).

Their k^{th}-order derivative sequences are defined as

$$U_n^{(k)} = \frac{d^k}{dx^k} U_n(x), \quad \text{and} \quad V_n^{(k)} = \frac{d^k}{dx^k} V_n(x).$$

Let us denote the complex numbers α, β, and γ, so that they satisfy:

$$\alpha + \beta + \gamma = x, \quad \alpha\beta + \alpha\gamma + \beta\gamma = 0, \quad \alpha\beta\gamma = 1. \tag{1.3}$$

2. POLYNOMIALS $U_n^{(k)}$ AND $V_n^{(k)}$

Using a known method, we can prove that the polynomials U_n and V_n possess generating functions as follows:

$$U(t) = t(1 - xt - t^3)^{-1} = \sum_{n=0}^{\infty} U_n t^n, \tag{2.1}$$

$$V(t) = (2 - xt)(1 - xt - t^3)^{-1} = \sum_{n=0}^{\infty} V_n t^n. \tag{2.2}$$

Differentiating both sides of (2.1), with respect to x, k-times, we get

$$U_k(t) = \frac{k! t^{k+1}}{(1 - xt - t^3)^{k+1}} = \sum_{n=0}^{\infty} U_n^{(k)} t^n. \tag{2.3}$$

Moreover, using induction on n, we can prove that the polynomials U_n and V_n satisfy the following relation

$$V_n = U_{n+1} + U_{n-2}, \quad n \geq 2. \tag{2.4}$$

Theorem 2.1: *Let k be a positive integer. Then it follows that*

$$U_k(t) = \frac{k!}{(\alpha A)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}}{(1 - \alpha t)^{k+1-i}} + \frac{k!}{(\beta M)^{k+1}} \sum_{i=0}^{k} \frac{b_{k,i}}{(1 - \beta t)^{k+1-i}}$$

$$+ \frac{k!}{(\gamma R)^{k+1}} \sum_{i=0}^{k} \frac{c_{k,i}}{(1 - \gamma t)^{k+1-i}}, \tag{2.5}$$

where

$$a_{k,i} = (-1)^i A^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} A^i B^{j-2l} C^l a_{k,i-j},$$

$$b_{k,i} = (-1)^i M^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} M^i N^{j-2l} P^l b_{k,i-j},$$

$$c_{k,i} = (-1)^i R^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} R^l S^{j-2l} T^l c_{k,i-j},$$

for $i = 1, \ldots, k$ *and*

$$A = A(\alpha) = \frac{\alpha^2(2\alpha - x) + 1}{\alpha^3},\ B = B(\alpha) = \frac{\alpha^2(x - \alpha) - 2}{\alpha^3},\ C = C(\alpha) = \frac{1}{\alpha^3},$$

$$M = A(\beta),\ N = N(\beta),\ P = C(\beta),\ R = A(\gamma),\ S = B(\gamma),\ T = C(\gamma).$$

Proof: From (1.3) and (2.3), we get

$$\frac{t^{k+1}}{(1 - xt - t^3)^{k+1}} = \sum_{i=0}^{k} \frac{A_{k,i}}{(1 - \alpha t)^{k+1-i}} + \sum_{i=0}^{k} \frac{B_{k,i}}{(1 - \beta t)^{k+1-i}} + \sum_{i=0}^{k} \frac{C_{k,i}}{(1 - \gamma t)^{k+1-i}}, \tag{2.6}$$

where $A_{k,i}, B_{k,i}$, and $C_{k,i}$ are independent of t.

Multiplying (2.6) by $\alpha^{k+1}(1 - \beta t)^{k+1}(1 - \gamma t)^{k+1}$, we get

$$\frac{(\alpha t)^{k+1}}{(1 - \alpha t)^{k+1}} = \alpha^{k+1}[A + B(1 - \alpha t) + C(1 - \alpha t)^2] \sum_{i=0}^{k+1} \frac{A_{k,i}}{(1 - \alpha t)^{k+1-i}} + \phi(t), \tag{2.7}$$

where $\phi(t)$ is an analytic function at the point $t = \alpha^{-1}$ (t is a complex variable and x is a real constant).

Since

$$\frac{(\alpha t)^{k+1}}{(1 - \alpha t)^{k+1}} = ((1 - \alpha t)^{-1} - 1)^{k+1},$$

from (2.7), it follows that

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i (1 - \alpha t)^{-(k+1-i)} =$$

$$\alpha^{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} A^{k+1-j} \sum_{l=0}^{j} \binom{j}{l} B^{j-l} C^l (1 - \alpha t)^{j+l} \sum_{i=0}^{k} \frac{A_{k,i}}{(1 - \alpha t)^{k+1-i}} + \phi(t).$$

Using the fact that the Laurent series [6] is unique at the point $t = \alpha^{-1}$ for the function $(\alpha t)^{k+1}(1 - \alpha t)^{-(k+1)}$, we can compare the coefficients of $(1 - \alpha t)^{-(k+1-i)} (i = 0, 1, \ldots, k)$ on both sides of the last equality. So, we get

$$\alpha^{k+1} \sum_{j=0}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} A^{k+1-j+l} B^{j-2l} C^l A_{k,i-j}, = (-1)^i \binom{k+1}{i}, \tag{2.8}$$

where $i = 0, 1, \ldots, k$, and

$$A = A(\alpha) = \frac{\alpha^2(2\alpha - x) + 1}{\alpha^3},\ B = B(\alpha) = \frac{\alpha^2(x - \alpha) - 2}{\alpha^3},\ C = C(\alpha) = \frac{1}{\alpha^3}.$$

Let us denote

$$a_{k,i-j} = \alpha^{k+1} A^{k+1+i-j} A_{k,i-j}.$$

Hence, from (2.8), we get

$$\sum_{j=0}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} A^l B^{j-2l} C^l a_{k,i-j} = (-1)^i A^i \binom{k+1}{i},$$

where $a_{k,0} = 1$.

¿From the last equality, for $j = 0$, it follows that

$$a_{k,i} = (-1)^i A^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} A^l B^{j-2l} C^l a_{k,i-j}. \tag{2.9}$$

In a similar way, we find that the coefficients $b_{k,i}$ and $c_{k,i}$ are given by

$$b_{k,i} = (-1)^i M^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} M^l N^{j-2l} P^l b_{k,i-j}; \tag{2.10}$$

$$c_{k,i} = (-1)^i R^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \binom{k+1}{j-l} \binom{j-l}{l} R^l S^{j-2l} T^l c_{k,i-j}, \tag{2.11}$$

where

$$b_{k,0} = c_{k,0} = 1,\ M = A(\beta),\ N = B(\beta),\ P = C(\beta),\ R = A(\gamma), S = B(\gamma),\ T = C(\gamma).$$

If we substitute (2.9), (2.10), and (2.11) in (2.3), we get

$$U_k(t) = \frac{k!}{(\alpha A)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}}{A^i (1 - \alpha t)^{k+1-i}} + \frac{k!}{(\beta M)^{k+1}} \sum_{i=0}^{k} \frac{b_{k,i}}{M^i (1 - \beta t)^{k+1-i}} +$$

$$\frac{k!}{(\gamma R)^{k+1}} \sum_{i=0}^{k} \frac{c_{k,i}}{R^i (1 - \gamma t)^{k+1-i}}. \quad \square$$

3. FURTHER INTERESTING IDENTITIES

Lemma 3.1: *Let n be a positive integer and r and m be nonnegative integers. Then*

$$\sum_{i=0}^{n} U_i = (U_{n+1} + U_n + U_{n-1} - 1)/x, \quad x \neq 0. \tag{3.1}$$

$$\sum_{i=0}^{n} V_i = (V_{n+1} + V_n + V_{n-1} - 1)/x, \quad x \neq 0. \tag{3.2}$$

$$\sum_{i=0}^{n} \binom{n}{i} x^i h_{r+2i} = h_{r+3n} \quad (h_n = U_n \text{ or } h_n = V_n). \tag{3.3}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} h_{r+3i} = (-1)^n x^n h_{r+2n} \quad (h_n = U_n \text{ or } h_n = V_n). \tag{3.4}$$

$$U_{m+n} = U_{m+1}U_n + U_m U_{n-2} + U_{m-1}U_{n-1}, \quad n \geq 2. \tag{3.5}$$

$$V_{m+n} = V_{m+1}U_n + V_m U_{n-2} + V_{m-1}U_{n-1}, \quad n \geq 2. \tag{3.6}$$

Proof: In the proof we use induction on n.
For $n = 1$ in (3.1), we get

$$U_0 + U_1 = \frac{1}{x}(U_2 + U_1 + U_0 - 1) = \frac{1}{x}(x + 1 + 0 - 1) = 1.$$

It follows that (3.1) holds for $n = 1$. Suppose that (3.1) holds for $n \geq 1$. Then, for $n + 1$, it follows that

$$\sum_{i=0}^{n+1} U_i = \sum_{i=0}^{n} U_i + U_{n+1}$$

$$= \frac{1}{x}(U_{n+1} + U_n + U_{n-1} + xU_{n+1} - 1) = \frac{1}{x}(U_{n+2} + U_{n+1} + U_n - 1).$$

Thus, we conclude that (3.1) holds for all $n \in N$.
Similarly, we can prove the equalities (3.2) and (3.3).
To prove (3.4), we also use induction on n. For $n = 1$ it follows that

$$\sum_{i=0}^{1} (-1)^i \frac{1}{i} h_{r+3i} = h_r + h_{r+3} = -x h_{r+2} \quad (\textit{by } (1.1) \textit{ and } (1.2)).$$

Hence, (3.4) is true for $n = 1$. Suppose that (3.4) is true for $n \geq 1$. Then, for $n = n+1$, we get

$$\begin{aligned}
(-1)^{n+1}x^{n+1}h_{r+2n+2} &= -x(-1)^n x^n h_{r+2+2n} = -x\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{r+2+3i} \\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}xh_{r+2+3i} = \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}\left(h_{r+3+3i} - h_{r+3i}\right) \\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}h_{r+3(i+1)} + \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}h_{r+3i} \\
&= \sum_{i=1}^{n+1}(-1)^{i}\binom{n}{i-1}h_{r+3i} + \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}h_{r+3i} \\
&= \sum_{i=1}^{n}(-1)^{i}\left(\binom{n}{n-1} + \binom{n}{i}\right)h_{r+3i} + (-1)^{n+1}h_{r+3(n+1)} + h_r \\
&= \sum_{i=0}^{n+1}(-1)^{i}\binom{n+1}{i}h_{r+3i}.
\end{aligned}$$

So, we conclude that (3.4) is true for all $n \in N$.

Equalities (3.5) and (3.6) can be proved using recurrence relations (1.1) and (1.2), and applying induction on n. □

Theorem 3.1: *Let n be a positive integer and k be a nonnegative integer.*

$$x\sum_{i=0}^{n}U_i^{(k)} = U_{n+1}^{(k)} + U_n^{(k)} + U_{n-1}^{(k)} - k\sum_{i=0}^{n}U_i^{(k-1)}, \quad x \neq 0; \tag{3.7}$$

$$x\sum_{i=0}^{n}V_i^{(k)} = V_{n+1}^{(k)} + V_n^{(k)} + V_{n-1}^{(k)} - k\sum_{i=0}^{n}V_i^{(k-1)}, \quad x \neq 0; \tag{3.8}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^i)^{(j)}h_{r+2i}^{(k-j)} = h_{r+3n}^{(k)}; \tag{3.9}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{r+3i}^{(k)} = (-1)^n\sum_{j=0}^{k}\binom{k}{j}(n-j+1)_j x^{n-j}h_{r+2n}^{(k-j)}, \tag{3.10}$$

where $h_n = U_n$ or $h_n = V_n$.

Proof: Equalities (3.7), (3.8), and (3.10), can be proved in a straightforward manner by differentiating the corresponding equalities (3.1), (3.2), and (3.4). Here, we prove (3.9).

If $k=0$, then (3.9) becomes

$$h_{r+3n}=\sum_{i=0}^{n}\binom{n}{i}x^{i}h_{r+2i}.$$

It follows that (3.4) is true. Suppose that (3.9) is true for $k\geq 0$. Then, for $k+1$, we get

$$\begin{aligned}
h_{r+3n}^{(k+1)} &= \frac{d}{dx}\left(h_{r+3n}^{(k)}\right)=\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\frac{d}{dx}\left((x^{i})^{(j)}h_{r+2i}^{(k-j)}\right)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\left((x^{i})^{(j+1)}h_{r+2i}^{(k-j)}+(x^{i})^{(j)}h_{r+2i}^{(k+1-j)}\right) \quad (j+1:=j)\\
&= \sum_{i=0}^{n}\sum_{j=1}^{k+1}\binom{n}{i}\binom{k}{j-1}(x^{i})^{(j)}h_{r+2i}^{(k+1-j)}+\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^{i})^{(j)}h_{r+2i}^{(k+1-j)}\\
&= \sum_{i=0}^{n}\sum_{j=1}^{k}\binom{n}{i}\left(\binom{k}{j-1}+\binom{k}{j}\right)(x^{i})^{(j)}h_{r+2i}^{(k+1-j)}+\sum_{i=0}^{n}\binom{n}{i}\binom{k}{k}(x^{i})^{(k+1)}h_{r+2i}\\
&+\sum_{i=0}^{n}\binom{n}{i}\binom{k+1}{0}x^{i}h_{r+2i}^{(k+1)}=\sum_{i=0}^{n}\sum_{j=0}^{k+1}\binom{n}{i}\binom{k+1}{j}(x^{i})^{(j)}h_{r+2i}^{(k+1-j)}. \quad \square
\end{aligned}$$

Theorem 3.2: *Let n be a positive integer and k be a nonnegative integer. Then*

$$h_{n}^{(k)}=xh_{n-1}^{(k)}+h_{n-3}^{(k)}+kh_{n-1}^{(k-1)},\ k\geq 0,\ (h_{n}=U_{n}\ \textit{or}\ h_{n}=V_{n}). \tag{3.11}$$

$$V_{n}^{(k)}=U_{n+1}^{(k)}+U_{n-2}^{(k)},\quad n\geq 2. \tag{3.12}$$

$$U_{n+m}^{(k)}=\sum_{i=0}^{k}\binom{k}{i}\left(U_{m+1}^{(k-i)}U_{n}^{(i)}+U_{m}^{(k-i)}U_{n-2}^{(i)}+U_{m-1}^{(k-i)}U_{n-1}^{(i)}\right). \tag{3.13}$$

$$V_{m+n}^{(k)}=\sum_{i=0}^{k}\binom{k}{i}\left(V_{m+1}^{(k-i)}U_{n}^{(i)}+V_{m}^{(k-i)}U_{n-2}^{(i)}+V_{m-1}^{(k-i)}U_{n-1}^{(i)}\right). \tag{3.14}$$

Proof: Equalities (3.11), (3.12), (3.13), and (3.14) can be proved by differentiating the corresponding equalities (1.1), (1.2), (2.4), (3.5), and (3.6). $\square$

Next, if we differentiate (2.2), with respect to x, k-times, we get

$$V_{k}(t)=\frac{k!t^{k}(1+t^{3})}{(1-xt-t^{3})^{k+1}}=\sum_{n=0}^{\infty}V_{n}^{(k)}t^{n}.$$

So, using $U_k(t)$ and $V_r(t)$, we can easily prove the following identities:

$$U_k(t)U_r(t) = \frac{k!r!}{(k+r+1)!}U_{k+r+1}(t);$$

$$U_k(t)V(t) = \frac{1}{k+1}(2t^{-1}-x)U_{k+1}(t);$$

$$V_k(t)V_r(t) = \frac{k!r!}{(k+r+1)!}(t^2+t^{-1})V_{k+r+1}(t)\ (k,r\geq 1);$$

$$U_k(t)V_r(t) = \frac{k!r!}{(k+r+1)!}V_{k+r+1}(t)\ (r,k\geq 1);$$

$$V_k(t)V(t) = \frac{1}{k+1}(2t^{-1}-x)V_{k+1}(t);$$

$$V(t)V(t) = (2t^{-1}-x)^2U_1(t).$$

Thus, comparing the coefficients of t^n both sides in the last equalities, we can prove the following theorem.

Theorem 3.3: *Let n be a positive integer and k be a nonnegative integer. Then*

$$\sum_{i=0}^{n} U_i^{(k)}U_{n-i}^{(r)} = \frac{k!r!}{(1+k+r)!}U_n^{(k+r+1)};$$

$$\sum_{i=0}^{n} U_i^{(k)}V_{n-i} = \frac{1}{k+1}\left(2U_{n+1}^{(k+1)} - xU_n^{(k+1)}\right) \quad (k,r\geq 1);$$

$$\sum_{i=0}^{n} V_i^{(k)}V_{n-i}^{(r)} = \frac{k!r!}{(k+r+1)!}\left(V_{n-2}^{(k+r+1)} + V_{n+1}^{(k+r+1)}\right);$$

$$\sum_{i=0}^{n} U_i^{(k)}V_{n-i}^{(r)} = \frac{k!r!}{(k+r+1)!}V_n^{(k+r+1)} \quad (r\geq 1);$$

$$\sum_{i=0}^{n} V_i^{(k)}V_{n-i} = \frac{1}{k+1}\left(2V_{n+1}^{(k+1)} - xV_n^{(k+1)}\right);$$

$$\sum_{i=0}^{n} V_iV_{n-i} = 4U_{n+2}^{(1)} - 4xU_{n+1}^{(1)} + x^2U_n^{(1)}.$$

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AMS Classification Numbers: 11B39, 26A24, 11B83

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