ON THE SPECTRUM OF REAL NUMBERS REVISITED

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1. INTRODUCTION AND MAIN RESULT

The floor function of a real number x, denoted $\lfloor x \rfloor$, is defined as the largest integer $n \leq x$. For a positive real number α one can associate a sequence of positive integers, called the spectrum of α, denoted $Spec(\alpha)$, which is constructed using the floor function as follows

$$Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots\} .$$

The spectra of real numbers have a variety of interesting properties, for example if α,$\beta \in \mathbb{R}^+$ with α irrational and $\alpha^{-1} + \beta^{-1} = 1$, then the sets $Spec(\alpha)$ and $Spec(\beta)$ form a disjoint covering of the natural numbers that, is $Spec(\alpha) \cap Spec(\beta) = \emptyset$ with $Spec(\alpha) \cup Spec(\beta) = \mathbb{N}$. When dealing with spectra, it is sometimes useful to know if a given finite sequence of integers represents the initial segment of the spectrum of a real number. Graham et al. [2] showed that a finite sequence $\{a_1, a_2, \dots, a_n\}$ was the first n terms of the spectrum of a real number if and only if the sequence was **nearly linear**, that is if for all $1 < k \leq n$ the following inequality holds

$$\max\{a_i + a_{k-i} : 1 \leq i < k\} \leq a_k \leq 1 + \min\{a_i + a_{k-i} : 1 \leq i < k\} . \tag{1}$$

Spectra of the form $\{\lfloor n\alpha \rfloor : n \in \mathbb{N}\}$ are usually referred to as the homogeneous spectrum of α. One can naturally extend this notion to the idea of a nonhomogeneous spectrum or β-nonhomogeneous spectra of α which are sequences of the form $\{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}$. Fraenkel et al. [1] showed that for a finite sequence $\{a_1, a_2, \dots, a_n\}$ there exists $\alpha, \beta \in \mathbb{R}^+$ such that $a_i = \lfloor i\alpha + \beta \rfloor$ if and only if

$$\max_{1 \leq i < r \leq n} \frac{a_r - a_{r-i} - 1}{i} < \min_{1 \leq i < r \leq n} \frac{a_r - a_{r-i} + 1}{i} . \tag{2}$$

In this note, we return to the case of homogeneous spectra and show that a sequence of positive integers $\{a_n\}$ represents the spectrum of a real number if and only if the following inequality holds for each n

$$\max_{1 \leq r \leq n} \left\{\frac{a_r}{r}\right\} < \min_{1 \leq r \leq n} \left\{\frac{a_r + 1}{r}\right\} . \tag{3}$$

As will be seen, the above characterisation in the form of an inequality reminiscent of (2), shall follow from an application of the bounded monotone convergence theorem for real sequences. We now state and prove the main result.

Theorem 1.1: *For a monotonic sequence of positive integers $\{a_n\}$, there exists an $\alpha \in \mathbb{R}^+$ such that $Spec(\alpha) = \{a_n\}_{n=1}^{\infty}$ if and only if for all $n \in \mathbb{N}$ (3) holds.*

Proof: We first note that the two intervals $[c_1, d_1)$ and $[c_2, d_2)$, where $c_i, d_i \in \mathbb{R}^+$, will have a non-empty intersection if and only if

$$\max\{c_1, c_2\} < \min\{d_1, d_2\} \ , \tag{4}$$

and, moreover, $[c_1, d_1) \cap [c_2, d_2) = [c, d)$, where $c = \max\{c_1, c_2\}$ and $d = \min\{d_1, d_2\}$. Now if there exists an $\alpha \in \mathbb{R}^+$ such that $\lfloor n\alpha \rfloor = a_n$ then $a_n \le n\alpha < a_n + 1$ and so $\alpha \in I_n = [\frac{a_n}{n}, \frac{a_n+1}{n})$ for all $n \in \mathbb{N}$. Hence $I_n \cap I_m \ne \emptyset$ for all $m, n \in \mathbb{N}$. Consequently from (4) one deduces the desired condition in (3). Suppose now the sequence $\{a_n\}$ satisfies (3). To produce the $\alpha \in \mathbb{R}^+$ having $Spec\,(\alpha) = \{a_n\}_{n=1}^{\infty}$ it will be necessary to examine the family of sets $\{A_n\}$ given by

$$A_n = \bigcap_{r=1}^{n} I_r \ .$$

We first show via induction that $A_n \ne \emptyset$ for all $n \in \mathbb{N}$ and that $A_n = [c_n, d_n)$ where

$$c_n = \max_{1 \le r \le n} \left\{ \frac{a_r}{r} \right\} \quad \text{and} \quad d_n = \min_{1 \le r \le n} \left\{ \frac{a_r + 1}{r} \right\} \ .$$

Clearly, from (4) the result holds for $n = 2$; assume the claim also holds for some $n = k \ge 2$ and consider the set $A_{k+1} = [c_k, d_k) \cap I_{k+1}$. If one supposes that $[c_k, d_k) \cap I_{k+1} = \emptyset$ then two possible cases present themselves; namely, either $d_k \le \frac{a_{k+1}}{k+1}$ or $\frac{a_{k+1}+1}{k+1} \le c_k$. Consider the former case and suppose $d_k = \frac{a_s+1}{s}$ for some $s \in \{1, 2, \ldots, k\}$; then we have the inequality

$$\frac{a_s}{s} < \frac{a_s + 1}{s} \le \frac{a_{k+1}}{k+1} < \frac{a_{k+1} + 1}{k+1} \ ,$$

from which it is immediately deduced that $\max\{\frac{a_s}{s}, \frac{a_{k+1}}{k+1}\} \ge \min\{\frac{a_s+1}{s}, \frac{a_{k+1}+1}{k+1}\}$. This is a contradiction to the assumed condition in (3). Similarly, in the latter case, if $c_k = \frac{a_p}{p}$ for some $p \in \{1, 2, \ldots, k\}$, then we have

$$\frac{a_{k+1}}{k+1} < \frac{a_{k+1} + 1}{k+1} \le \frac{a_p}{p} < \frac{a_p + 1}{p} \ ,$$

from which again we deduce the contradictory inequality $\max\{\frac{a_{k+1}}{k+1}, \frac{a_p}{p}\} \ge \min\{\frac{a_{k+1}+1}{k+1}, \frac{a_{p+1}}{p}\}$. Hence, one must have $[c_k, d_k) \cap I_{k+1} \ne \emptyset$. Furthermore, by writing $A_{k+1} = [c_{k+1}, d_{k+1})$, we obtain from the inductive assumption that

$$c_{k+1} = \max\left\{ c_k, \frac{a_{k+1}}{k+1} \right\} = \max_{1 \le r \le k+1} \left\{ \frac{a_r}{r} \right\}$$

and

$$d_{k+1} = \min\left\{ d_k, \frac{a_{k+1} + 1}{k+1} \right\} = \min_{1 \le r \le k+1} \left\{ \frac{a_r + 1}{r} \right\} \ .$$

Now, by construction, the sequences $\{c_n\}$ and $\{d_n\}$ are, respectively, monotone increasing and decreasing. Moreover, c_n is bounded above by $a_1 + 1$, whereas b_n is bounded below by a_1, as $A_n \subseteq I_1$. Hence from the bounded monotone convergence theorem the two sequences must converge to a finite limit, denoted here by c and d, respectively. In addition, as $A_n \subseteq I_n$, one also has $|d_n - c_n| < \frac{1}{n}$, from which it is deduced upon taking limits as $n \to \infty$ that $c = d = \alpha$, say. Since $c_n \le \alpha \le d_n$, we have $\alpha \in A_n$ for all $n \in \mathbb{N}$, and so

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bigcap_{r=1}^{n} I_r = \bigcap_{n=1}^{\infty} I_n \neq \emptyset \ .$$

Furthermore, $\cap_{n=1}^{\infty} A_n = \{\alpha\}$, since if $x < \alpha$ or $y > \alpha$ then by the monotonicity of the above sequences there must exist an $M \in \mathbb{N}$ such that $x < c_n \le \alpha < d_n < y$ for $n > M$, and so $x, y \notin A_n$ for $n > M$. Consequently, α must have the property that $a_n \le n\alpha < a_n + 1$, and so $\lfloor n\alpha \rfloor = a_n$ for all $n \in \mathbb{N}$ as required. □

To close it is easy to demonstrate that (3) implies that the sequence $\{a_n\}$ is nearly linear. Indeed, if $\{a_n\}$ satisfies (3) then by Theorem 1.1 there exists a unique $\alpha \in \mathbb{R}^+$ such that $a_k = \lfloor k\alpha \rfloor$. Now if $1 \le i < k \le n$, then by definition of $\lfloor \cdot \rfloor$ we have $\lfloor i\alpha \rfloor \le i\alpha < \lfloor i\alpha \rfloor + 1$ and $\lfloor (k-i)\alpha \rfloor \le (k-i)\alpha < \lfloor (k-i)\alpha \rfloor + 1$. Adding the previous inequalities together gives

$$\lfloor i\alpha \rfloor + \lfloor (k-i)\alpha \rfloor \le k\alpha < \lfloor i\alpha \rfloor + \lfloor (k-i)\alpha \rfloor + 2 \ ,$$

from which it is deduced, as $\lfloor k\alpha \rfloor \in \mathbb{N}$, that

$$\lfloor i\alpha \rfloor + \lfloor (k-i)\alpha \rfloor \le \lfloor k\alpha \rfloor \le \lfloor i\alpha \rfloor + \lfloor (k-i)\alpha \rfloor + 1 \ .$$

Hence (1) must hold and so $\{a_n\}$ is a nearly linear sequence.

REFERENCES

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