HEPTAGONAL NUMBERS IN THE ASSOCIATED PELL SEQUENCE AND DIOPHANTINE EQUATIONS $x^2(5x-3)^2 = 8y^2 \pm 4$

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1. INTRODUCTION

We denote the m^{th} g-gonal number by

$$\mathcal{G}_{m,g} = m\{(g-2)m - (g-4)\}/2 \quad \text{(see [1])}.$$

If m is positive and $g = 3, 4, 5, 6, 7, 8, \ldots$, etc., then the number $\mathcal{G}_{m,g}$ is triangular, square, pentagonal, hexagonal, heptagonal and octagonal etc., respectively. Finding the numbers common to any two infinite sequences is one of the problems in Number Theory. Several papers (See [2] to [15]) have appeared identifying the numbers $\mathcal{G}_{m,g}$ (for $g = 3, 4, 5$ and 7) in the sequences $\{F_n\}, \{L_n\}, \{P_n\}$ and $\{Q_n\}$ (the Fibonacci, Lucas, Pell and Associated Pell sequences respectively). We will summarize these results in Table A, including the present result that 1, 7 and 99 are the only generalized heptagonal numbers in the associated Pell sequence $\{Q_n\}$ defined by

$$Q_0 = Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for any integer } n.$$

This result also solves the two Diophantine equations in the title.

Sequences [12]		**Triangular (A000217)**	**Square (A000290)**	**Pentagonal (A000326)**	**Heptagonal (A000566)**
Fibonacci	**by**	Ming Luo [4]	J.H.E. Cohn [2]	Ming Luo [6]	B. Srinivasa Rao [14]
$\{\mathbf{F_n}\}$	n	0, ±1, 2, 4, 8, 10	0, ±1, 2, 12	0, ±1, 2, ±5	0, ±1, 2, ±7, ±9, 10
(A000045)	$\mathbf{F_n}$	0, 1, 3, 21, 55	0, 1, 144	0, 1, 5	0, 1, 13, 34, 55
Lucas	**by**	Ming Luo [5]	J.H.E. Cohn [2]	Ming Luo [7]	B. Srinivasa Rao [13]
$\{\mathbf{L_n}\}$	n	1, ±2	1, 3	0, 1, ±4	1, 3, ±4, ±6
(A000032)	$\mathbf{L_n}$	1, 3	1, 4	2, 1, 7	1, 4, 7, 18
Pell	**by**	Wayne McDaniel [8]	Katayama, S.I. & Katayama, S.G. [3]	V.S.R. Prasad & B. Srinivasa Rao [10]	B. Srinivasa Rao [15]
$\{\mathbf{P_n}\}$	n	0, ±1	0, ±1, ±7	0, ±1, 2, ±3, 4, 6	0, ±1, 6
(A000129)	$\mathbf{P_n}$	0, 1	0, 1, 169	0, 1, 2, 5, 12, 70	0, 1, 70
Associated Pell	**by**	V.S.R. Prasad & B. Srinivasa Rao [11]	Katayama, S.I. & Katayama, S.G. [3]	V.S.R. Prasad & B. Srinivasa Rao [9]	Present Result
$\{\mathbf{Q_n}\}$	n	0, 1, ±2	0, 1	0, 1, 3	0, 1, 3, ±6
(A001333)	$\mathbf{Q_n}$	1, 3	1	1, 7	1, 7, 99

Table A.

In the above table, by a polygonal number we mean a generalized polygonal number (with m any integer). Further, each cell where a column and a row meet represents the numbers common to both the corresponding sequences named after the person who identified them.

2. MAIN THEOREM

We need the following well known properties of $\{P_n\}$ and $\{Q_n\}$: For all integers k, m and n.

$$\left.\begin{array}{c} P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\alpha^n+\beta^n}{2} \\ \text{where } \alpha = 1+\sqrt{2} \text{ and } \beta = 1-\sqrt{2} \end{array}\right\} \tag{1}$$

$$P_{-n} = (-1)^{n+1}P_n \text{ and } Q_{-n} = (-1)^n Q_n \tag{2}$$

$$Q_n^2 = 2P_n^2 + (-1)^n \tag{3}$$

$$Q_{m+n} = 2Q_mQ_n - (-1)^n Q_{m-n} \tag{4}$$

$$3|P_n \text{ iff } 4|n \text{ and } 3|Q_n \text{ iff } n \equiv 2 \pmod 4 \tag{5}$$

$$9|P_n \text{ iff } 12|n \text{ and } 9|Q_n \text{ iff } n \equiv 6 \pmod{12} \tag{6}$$

If m is even, then (see [9])

$$Q_{n+2km} \equiv (-1)^k Q_n \pmod{Q_m} \tag{7}$$

Theorem: (a) Q_n is a generalized heptagonal number only for $n = 0, 1, 3$ or ± 6;
and (b) Q_n is a heptagonal number only for $n = 0, 1$ or 3.

Proof: (a) Case 1: Suppose $n \equiv 0, 1, 3, \pm 6 \pmod{600}$.

Then it is sufficient to prove that $40Q_n + 9$ is a perfect square if and only if $n = 0, 1, 3, \pm 6$. To prove this, we adopt the following procedure which enables us to tabulate the corresponding values reducing repetition and space.

Suppose $n \equiv \varepsilon \pmod N$ and $n \neq \varepsilon$. Then n can be written as $n = 2 \cdot \delta \cdot 2^\theta \cdot g + \varepsilon$, where $\theta \geq \gamma$ and $2 \not| g$. Furthermore, $n = 2km + \varepsilon$, where k is odd and m is even.

Now, using (7), we get

$$40Q_n + 9 = 40Q_{2km+\varepsilon} + 9 \equiv 40(-1)^k Q_\varepsilon + 9 \pmod{Q_m}.$$

Therefore, the Jacobi symbol

$$\left(\frac{40Q_n + 9}{Q_m}\right) = \left(\frac{-40Q_\epsilon + 9}{Q_m}\right) = \left(\frac{Q_m}{M}\right). \tag{8}$$

But modulo M, $\{Q_n\}$ is periodic with period P (here if $n \equiv 2 \pmod 4$, then we choose P as a multiple of 4 so that $3 \not| Q_m$). Now, since for $\theta \geq \gamma, 2^{\theta+s} \equiv 2^\theta \pmod P$, choosing $m = \mu \cdot 2^\theta$ if $\theta \equiv \zeta \pmod s$ and $m = 2^\theta$ otherwise, we have $m \equiv c \pmod P$ and $\left(\frac{Q_m}{M}\right) = -1$, for all values of m. From (8), it follows that $\left(\frac{40Q_n+9}{Q_m}\right) = -1$, for $n \neq \varepsilon$. For each value of ε, the corresponding values are tabulated in this way (Table B).

<table>
<tr><th>ε</th><th>N</th><th>δ</th><th>γ</th><th>s</th><th>M</th><th>P</th><th>μ</th><th>ζ (mod s)</th><th>c (mod P)</th></tr>
<tr><td>0, 1</td><td>20</td><td>5</td><td>1</td><td>4</td><td>31</td><td>30</td><td>5</td><td>0, 3</td><td>2, 4, ±10.</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">100</td><td rowspan="2">25</td><td rowspan="2">1</td><td rowspan="2">36</td><td rowspan="2">271</td><td rowspan="2">270</td><td>25</td><td>6, 7, 15, 24.</td><td rowspan="2">2, ±10, ±20, 32, 34, ±40, 70, 76, ±80, 94, 106, 140, 152, 154, 158, 166, 182, 184, 188, 196, 212, 242, 248 256.</td></tr>
<tr><td>5</td><td>0, 2, 3, 4, 10, 12, 13, ±17, 18, 21, 22, 31, 35.</td></tr>
<tr><td rowspan="3">±6</td><td rowspan="3">600</td><td rowspan="3">75</td><td rowspan="3">2</td><td rowspan="3">18</td><td rowspan="3">439</td><td rowspan="3">2·438 =876</td><td>25</td><td>3, 7.</td><td rowspan="3">4, 16, 32, 64, 192, 200, 220, 256, 296, 332, 440, 512, 548, 572, 616, 664. 712, 740.</td></tr>
<tr><td>5</td><td>10, 12, 13.</td></tr>
<tr><td>3</td><td>15.</td></tr>
</table>

Table B.

Since L.C.M. of (20, 100, 600)=600, the first part of the theorem follows for $n \equiv 0, 1, 3$ or $\pm 6 \pmod{600}$.

Case 2: Suppose $n \not\equiv 0, 1, 3$ or $\pm 6 \pmod{600}$. Step by step we proceed to eliminate certain integers n congruent modulo 600 for which $40Q_n + 9$ is not a square. In each step we choose an integer m such that the period k (of the sequence $\{Q_n\}$mod m) is a divisor of 600 and thereby eliminate certain residue classes modulo k. For example.

Mod 41: The sequence $\{Q_n\}$ mod 41 has period 10. We can eliminate $n \equiv \pm 2 \pmod{10}$, since $40Q_n + 9 \equiv 6 \pmod{41}$ and 6 is a quadratic nonresidue modulo 41. There remain $n \equiv 0, 1, 3, 4, 5, 6, 7$ or $9 \pmod{10}$.

Similarly we can eliminate the remaining values of n. We tabulate them in the following way (Table C) which proves part (a) of the theorem completely.

<table>
<tr><th>Period k</th><th>Modulus m</th><th>Required values of n where $\left(\frac{40Q_n+9}{m}\right)=-1$</th><th>Left out values of n (mod t) where t is a positive integer</th></tr>
<tr><td>10</td><td>41</td><td>±2.</td><td>0, ±1, ±3, ±4, or 5 (mod 10)</td></tr>
<tr><td>20</td><td>29</td><td>10, 11, 13, 17 and 19</td><td>0, 1, 3, ±4, ±5, ±6, 7 or 9 (mod 20)</td></tr>
<tr><td rowspan="2">100</td><td>1549</td><td>15, ±16, ±20, 21, 29, 35, ±46, 55, 63, 69, 81, 87 and 95,</td><td rowspan="2">0, 1, 3, ±6, 9, ±14, 23, ±24, ±25, ±26, 27, ±36, ±40, 41, 47, 49, 61, 67, 83 or 89 (mod 100)</td></tr>
<tr><td>29201</td><td>±4, 5, 7, ±34, 43, ±44, 45, 65 and 85.</td></tr>
<tr><td>30</td><td>31</td><td>±5, 7, ±9, 11 and 17.</td><td rowspan="6">0, 1, 3, ±6, ±75 or 183 (mod 300)</td></tr>
<tr><td>60</td><td>269</td><td>43 and 49.</td></tr>
<tr><td rowspan="3">150</td><td>751</td><td>±14, ±24, 27, ±36, ±40, ±44, 49, ±56, ±61, ±64, ±74, 117, 133, 139, and 147.</td></tr>
<tr><td>151</td><td>±26, ±50, 59, ±60, 73, 83, 123 and 149.</td></tr>
<tr><td>1201</td><td>±10, 23, 53 and 91.</td></tr>
<tr><td>600</td><td>9001</td><td>±75, 183, ±225, ±294, 300, 301, 303 and 483.</td><td>0, 1, 3, or ±6 (mod 600)</td></tr>
</table>

Table C.

For part (b), since, an integer N is heptagonal if and only if $40N+9=(10\cdot m-3)^2$ where m is a positive integer, we have the following table which proves the theorem.

$\boldsymbol{n}$	0	1	3	±6
$\boldsymbol{Q_n}$	1	1	7	99
$\boldsymbol{40Q_n+9}$	7^2	7^2	17^2	63^2
$\boldsymbol{m}$	1	1	2	-6
$\boldsymbol{P_n}$	0	1	5	±70

Table D.

If d is a positive integer which is not a perfect square it is well known that $x^2-dy^2=\pm1$ is called the Pell's equation and that if $x_1+y_1\sqrt{d}$ is the fundamental solution of it (that is, x_1 and y_1 are least positive integers), then $x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n$ is also a solution of the same equation; and conversely every solution of it is of this form. Now by (3), it follows that

$$Q_{2n}+\sqrt{2}P_{2n} \text{ is a solution of } x^2-2y^2=1,$$

while

$$Q_{2n+1}+\sqrt{2}P_{2n+1} \text{ is a solution of } x^2-2y^2=-1.$$

Therefore, by Table D and the Theorem, the two corollaries follows.

Corollary 1: The solution set of the Diophantine equation $x^2(5x-3)^2=8y^2-4$ is $\{(1,\pm1),(2,\pm5)\}$.

Corollary 2: The solution set of the Diophantine equation $x^2(5x-3)^2 = 8y^2 + 4$ is $\{(1,0), (-6, \pm 70)\}$.

REFERENCES

[1] Shiro Ando. "A Note on the Polygonal Numbers." *The Fibonacci Quarterly* **19** (1981): 180-183.

[2] J. H. E. Cohn. "Lucas and Fibonacci Numbers and Some Diophantine Equations." *Proc. Glasgow Math. Assn.* **7** (1965): 24-28.

[3] Shin-Ichi Katayama and Shigeru Katayama. "Fibonacci, Lucas and Pell Numbers and Class Numbers of Bicyclic Biquadratic Fields." *Math. Japonica* **42.1** (1995): 121-126.

[4] Ming Luo. "On Triangular Fibonacci Numbers." *The Fibonacci Quarterly* **27.2** (1989): 98-108.

[5] Ming Luo. "On Triangular Lucas Numbers." *Applications of Fibonacci Numbers.* Volume 4. Ed. G.E. Bergum, et. al. Kluwer Academic Pub., 1991, pp. 231-240.

[6] Ming Luo. "Pentagonal Numbers in the Fibonacci Sequence." *Applications of Fibonacci Numbers.* Volume 6. Ed. G.E. Bergum, et. al. Kluwer Academic Pub., 1994.

[7] Ming Luo. "Pentagonal Numbers in the Lucas Sequence." *Portugaliae Mathematica* **53.3** (1996): 325-329.

[8] W. L. McDaniel. "Triangular Numbers in the Pell Sequence." *The Fibonacci Quarterly* **34.2** (1996): 105-107.

[9] V. Siva Rama Prasad and B. Srinivasa Rao. "Pentagonal Numbers in the Associated Pell Sequence and Diophantine Equations $x^2(3x-1)^2 = 8y^2 \pm 4$." *The Fibonacci Quarterly* **39.4** (2001): 299-303.

[10] V. Siva Rama Prasad and B. Srinivasa Rao. "Pentagonal Numbers in the Pell Sequence and Diophantine Equations $2x^2 = y^2(3y-1)^2 \pm 2$." *The Fibonacci Quarterly* **40.3** (2002): 233-241.

[11] V. Siva Rama Prasad and B. Srinivasa Rao. "Triangular Numbers in the Associated Pell Sequence and Diophantine Equations $x^2(x+1)^2 = 8y^2 \pm 4$." *Indian J. Pure Appl. Math* **33.11** (2002): 1643-1648.

[12] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/sequences, A00032, A000129, A000045, A000217, A000290, A000326, A000566, A001333.

[13] B. Srinivasa Rao. "Heptagonal Numbers in the Lucas Sequence and Diophantine Equations $x^2(5x-3)^2 = 20y^2 \pm 16$." *The Fibonacci Quarterly* **40.4** (2002): 319-322.

[14] B. Srinivasa Rao. "Heptagonal Numbers in the Fibonacci Sequence and Diophantine Equations $4x^2 = 5y^2(5y-3)^2 \pm 16$." *The Fibonacci Quarterly* **41.5** (2003): 414-420.

[15] B. Srinivasa Rao. "Heptagonal Numbers in the Pell Sequence and Diophantine Equations $2x^2 = y^2(5y-3)^2 \pm 2$." *The Fibonacci Quarterly* **43.3** (2005): 194-201.

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