NON-TRIVIAL INTERTWINED SECOND-ORDER RECURRENCE RELATIONS

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1. INTRODUCTION

There are essentially seven different non–trivial systems of three intertwined second–order recurrence relations, as shown in [2], pp. 30–37, but the solutions are not given there. The object of this note is to give explicit solutions to all seven systems.

It appears that this area of study was initiated by K. Atanassov [1], with further contributions by K. Atanassov, J. Hlebarova and S. Mihov [3], and W. Spickerman, R. Creech and R. Joyner [4], [5]. In the last of these, solutions are in terms of recurrent sequences of order 6. Our solutions are in terms of Fibonacci numbers and various recurrences of orders 3 and 4 that grow more slowly than the Fibonacci numbers.

The seven systems and their solutions are given in §2.

We introduce the following six recurrent sequences. Explicit formulas are given in §3 for all these sequences.

The derivations of all our results are given in §4.

Let G_n, H_n, J_n, K_n, L_n and M_n be defined by
$G_0 = 1,\ G_1 = 2,\ G_2 = 2,\ G_3 = 1$
and for $n \geq 0$,

$$G_{n+4} = 2G_{n+3} - 2G_{n+2} + G_{n+1} - G_n,$$

$H_0 = 1,\ H_1 = -1,\ H_2 = 2,\ H_3 = -2$
and for $n \geq 0$,

$$H_{n+4} = -H_{n+3} + H_{n+2} + H_{n+1} - H_n,$$

$J_0 = 1,\ J_1 = -1,\ J_2 = -1,\ J_3 = 1$
and for $n \geq 0$,

$$J_{n+4} = -J_{n+3} - 2J_{n+2} - 2J_{n+1} - J_n,$$

$K_0 = 1,\ K_1 = -1,\ K_2 = 1$
and for $n \geq 0$,

$$K_{n+3} = -K_{n+2} - K_n,$$

$L_0 = 1,\ L_1 = 0,\ L_2 = -1$
and for $n \geq 0$,

$$L_{n+3} = -L_{n+1} + L_n$$

and
$M_0 = 1,\ M_1 = 0,\ M_2 = 0,\ M_3 = 0$
and for $n \geq 0$,

$$M_n = -M_{n-4}.$$

2. THE SYSTEMS AND THEIR SOLUTIONS

The seven non-trivial systems of intertwined second-order recurrences, together with their solutions, are as follows. Note that in each case we have multiplied through by 3 so as to clear fractions.

$$\begin{aligned} a_{n+2} &= a_{n+1} + b_n, \\ b_{n+2} &= b_{n+1} + c_n, \\ c_{n+2} &= c_{n+1} + a_n. \end{aligned} \tag{1}$$

$$\begin{aligned} 3a_n &= (F_{n-1} + 2G_n - 4G_{n-1} + 3G_{n-2} - G_{n-3})\, a_0 + (F_n + 2G_{n-1} - 2G_{n-2} + G_{n-3})\, a_1 \\ &\quad + (F_{n-1} - G_n + 2G_{n-1} - G_{n-3})\, b_0 + (F_n - G_{n-1} + G_{n-2} + G_{n-3})\, b_1 \\ &\quad + (F_{n-1} - G_n + 2G_{n-1} - 3G_{n-2} + 2G_{n-3})\, c_0 + (F_n - G_{n-1} + G_{n-2} - 2G_{n-3})\, c_1, \\ 3b_n &= (F_{n-1} - G_n + 2G_{n-1} - 3G_{n-2} + 2G_{n-3})\, a_0 + (F_n - G_{n-1} + G_{n-2} - 2G_{n-3})\, a_1 \\ &\quad + (F_{n-1} + 2G_n - 4G_{n-1} + 3G_{n-2} - G_{n-3})\, b_0 + (F_n + 2G_{n-1} - 2G_{n-2} + G_{n-3})\, b_1 \\ &\quad + (F_{n-1} - G_n + 2G_{n-1} - G_{n-3})\, c_0 + (F_n - G_{n-1} + G_{n-2} + G_{n-3})\, c_1, \\ 3c_n &= (F_{n-1} - G_n + 2G_{n-1} - G_{n-3})\, a_0 + (F_n - G_{n-1} + G_{n-2} + G_{n-3})\, a_1 \\ &\quad + (F_{n-1} - G_n + 2G_{n-1} - 3G_{n-2} + 2G_{n-3})\, b_0 + (F_n - G_{n-1} + G_{n-2} - 2G_{n-3})\, b_1 \\ &\quad + (F_{n-1} + 2G_n - 4G_{n-1} + 3G_{n-2} - G_{n-3})\, c_0 + (F_n + 2G_{n-1} - 2G_{n-2} + G_{n-3})\, c_1. \end{aligned}$$

$$\begin{aligned} a_{n+2} &= b_{n+1} + a_n, \\ b_{n+2} &= c_{n+1} + b_n, \\ c_{n+2} &= a_{n+1} + c_n. \end{aligned} \tag{2}$$

$$\begin{aligned} 3a_n &= (F_{n-1} + 2H_n + 2H_{n-1} - H_{n-3})\, a_0 + (F_n + 2H_{n-1} + H_{n-2} - 2H_{n-3})\, a_1 \\ &\quad + (F_{n-1} - H_n - H_{n-1} + 2H_{n-3})\, b_0 + (F_n - H_{n-1} + H_{n-2} + H_{n-3})\, b_1 \\ &\quad + (F_{n-1} - H_n - H_{n-1} - H_{n-3})\, c_0 + (F_n - H_{n-1} - 2H_{n-2} + H_{n-3})\, c_1, \\ 3b_n &= (F_{n-1} - H_n - H_{n-1} - H_{n-3})\, a_0 + (F_n - H_{n-1} - 2H_{n-2} + H_{n-3})\, a_1 \\ &\quad + (F_{n-1} + 2H_n + 2H_{n-1} - H_{n-3})\, b_0 + (F_n + 2H_{n-1} + H_{n-2} - 2H_{n-3})\, b_1 \\ &\quad + (F_{n-1} - H_n - H_{n-1} + 2H_{n-3})\, c_0 + (F_n - H_{n-1} + H_{n-2} + H_{n-3})\, c_1, \\ 3c_n &= (F_{n-1} - H_n - H_{n-1} + 2H_{n-3})\, a_0 + (F_n - H_{n-1} + H_{n-2} + H_{n-3})\, a_1 \\ &\quad + (F_{n-1} - H_n - H_{n-1} - H_{n-3})\, b_0 + (F_n - H_{n-1} - 2H_{n-2} + H_{n-3})\, b_1 \\ &\quad + (F_{n-1} + 2H_n + 2H_{n-1} - H_{n-3})\, c_0 + (F_n + 2H_{n-1} + H_{n-2} - 2H_{n-3})\, c_1. \end{aligned}$$

$$\begin{aligned} a_{n+2} &= b_{n+1} + b_n, \\ b_{n+2} &= c_{n+1} + c_n, \\ c_{n+2} &= a_{n+1} + a_n. \end{aligned} \tag{3}$$

$$\begin{aligned}
3a_n &= (F_{n-1} + 2J_n + 2J_{n-1} + 3J_{n-2} + 2J_{n-3})\, a_0 + (F_n + 2J_{n-1} + J_{n-2} + J_{n-3})\, a_1 \\
&\quad + (F_n - J_{n-1} + J_{n-2} + J_{n-3})\, b_0 + (F_{n-1} - J_n - J_{n-1} - J_{n-3})\, b_1 \\
&\quad + (F_{n-1} - J_n - J_{n-1} - 3J_{n-2} - J_{n-3})\, c_0 + (F_n - J_{n-1} - 2J_{n-2} - 2J_{n-3})\, c_1, \\
3b_n &= (F_{n-1} - J_n - J_{n-1} - 3J_{n-2} - J_{n-3})\, a_0 + (F_n - J_{n-1} - 2J_{n-2} - 2J_{n-3})\, a_1 \\
&\quad + (F_{n-1} + 2J_n + 2J_{n-1} + 3J_{n-2} + 2J_{n-3})\, b_0 + (F_n + 2J_{n-1} + J_{n-2} + J_{n-3})\, b_1 \\
&\quad + (F_n - J_{n-1} + J_{n-2} + J_{n-3})\, c_0 + (F_{n-1} - J_n - J_{n-1} - J_{n-3})\, c_1, \\
3c_n &= (F_n - J_{n-1} + J_{n-2} + J_{n-3})\, a_0 + (F_{n-1} - J_n - J_{n-1} - J_{n-3})\, a_1 \\
&\quad + (F_{n-1} - J_n - J_{n-1} - 3J_{n-2} - J_{n-3})\, b_0 + (F_n - J_{n-1} - 2J_{n-2} - 2J_{n-3})\, b_1 \\
&\quad + (F_{n-1} + 2J_n + 2J_{n-1} + 3J_{n-2} + 2J_{n-3})\, c_0 + (F_n + 2J_{n-1} + J_{n-2} + J_{n-3})\, c_1.
\end{aligned}$$

$$\begin{aligned}
a_{n+2} &= b_{n+1} + c_n, \\
b_{n+2} &= c_{n+1} + a_n, \\
c_{n+2} &= a_{n+1} + b_n.
\end{aligned} \tag{4}$$

$$\begin{aligned}
3a_n &= \left(1 + 2\cos\frac{2n\pi}{3}\right) F_{n-1}a_0 + \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_n a_1 \\
&\quad + \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_{n-1}b_0 + \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_n b_1 \\
&\quad + \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_{n-1}c_0 + \left(1 + 2\cos\frac{2n\pi}{3}\right) F_n c_1, \\
3b_n &= \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_{n-1}a_0 + \left(1 + 2\cos\frac{2n\pi}{3}\right) F_n a_1 \\
&\quad + \left(1 + 2\cos\frac{2n\pi}{3}\right) F_{n-1}b_0 + \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_n b_1 \\
&\quad + \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_{n-1}c_0 + \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_n c_1, \\
3c_n &= \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_{n-1}a_0 + \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_n a_1 \\
&\quad + \left(1 + 2\cos\frac{2(n+1)\pi}{3}\right) F_{n-1}b_0 + \left(1 + 2\cos\frac{2n\pi}{3}\right) F_n b_1 \\
&\quad + \left(1 + 2\cos\frac{2n\pi}{3}\right) F_{n-1}c_0 + \left(1 + 2\cos\frac{2(n-1)\pi}{3}\right) F_n c_1.
\end{aligned}$$

$$\begin{aligned}
a_{n+2} &= a_{n+1} + b_n, \\
b_{n+2} &= c_{n+1} + a_n, \\
c_{n+2} &= b_{n+1} + c_n.
\end{aligned} \tag{5}$$

$$\begin{aligned}
3a_n &= (F_{n-1} + 2K_n + 2K_{n-1} - K_{n-2})\, a_0 + (F_n - K_n + 2K_{n-2} + 1)\, a_1 \\
&\quad + (F_{n-1} - 2K_n - 3K_{n-1} + 1)\, b_0 + (F_n - K_{n-1} - 2K_{n-2})\, b_1 \\
&\quad + (F_{n-1} + K_{n-1} + K_{n-2} - 1)\, c_0 + (F_n + K_n + K_{n-1} - 1)\, c_1, \\
3b_n &= (F_{n-1} - K_n - K_{n-1} + 2K_{n-2})\, a_0 + (F_n - K_{n-1} - 2K_{n-2})\, a_1 \\
&\quad + (F_{n-1} + 2K_n + 2K_{n-1} - K_{n-2})\, b_0 + (F_n + 2K_{n-1} + K_{n-2})\, b_1 \\
&\quad + (F_{n-1} - K_n - K_{n-1} - K_{n-2})\, c_0 + (F_n - K_{n-1} + K_{n-2})\, c_1, \\
3c_n &= (F_{n-1} - K_n - K_{n-1} - K_{n-2})\, a_0 + (F_n + K_n + K_{n-1} - 1)\, a_1 \\
&\quad + (F_{n-1} + K_{n-1} + K_{n-2} - 1)\, b_0 + (F_n - K_{n-1} + K_{n-2})\, b_1 \\
&\quad + (F_{n-1} + K_n + 1)\, c_0 + (F_n - K_n - K_{n-2} + 1)\, c_1.
\end{aligned}$$

$$\begin{aligned}
a_{n+2} &= a_{n+1} + b_n, \\
b_{n+2} &= c_{n+1} + c_n, \\
c_{n+2} &= b_{n+1} + a_n.
\end{aligned} \tag{6}$$

$$\begin{aligned}
3a_n &= (F_{n-1} + 2M_n - M_{n-2} - M_{n-3})\, a_0 + (F_n + 2M_{n-1} + 2M_{n-2} + M_{n-3})\, a_1 \\
&\quad + (F_{n-1} - M_n + 2M_{n-2} + 2M_{n-3})\, b_0 + (F_n - M_{n-1} - M_{n-2} + M_{n-3})\, b_1 \\
&\quad + (F_{n-1} - M_n - M_{n-2} - M_{n-3})\, c_0 + (F_n - M_{n-1} - M_{n-2} - 2M_{n-3})\, c_1, \\
3b_n &= (F_{n-1} - M_n - M_{n-2} + 2M_{n-3})\, a_0 + (F_n - M_{n-1} - M_{n-2} - 2M_{n-3})\, a_1 \\
&\quad + (F_{n-1} + 2M_n - M_{n-2} - M_{n-3})\, b_0 + (F_n + 2M_{n-1} - M_{n-2} + M_{n-3})\, b_1 \\
&\quad + (F_{n-1} - M_n + 2M_{n-2} - M_{n-3})\, c_0 + (F_n - M_{n-1} + 2M_{n-2} + M_{n-3})\, c_1, \\
3c_n &= (F_{n-1} - M_n + 2M_{n-2} - M_{n-3})\, a_0 + (F_n - M_{n-1} - M_{n-2} + M_{n-3})\, a_1 \\
&\quad + (F_{n-1} - M_n - M_{n-2} - M_{n-3})\, b_0 + (F_n - M_{n-1} + 2M_{n-2} - 2M_{n-3})\, b_1 \\
&\quad + (F_{n-1} + 2M_n - M_{n-2} + 2M_{n-3})\, c_0 + (F_n + 2M_{n-1} - M_{n-2} + M_{n-3})\, c_1.
\end{aligned}$$

$$\begin{aligned}
a_{n+2} &= b_{n+1} + a_n, \\
b_{n+2} &= c_{n+1} + c_n, \\
c_{n+2} &= a_{n+1} + b_n.
\end{aligned} \tag{7}$$

$$\begin{aligned}
3a_n &= (F_{n-1} + L_n + L_{n-1} + 2L_{n-2} + (-1)^n)\, a_0 + (F_n + L_n + L_{n-1} + L_{n-2} - (-1)^n)\, a_1 \\
&\quad + (F_{n-1} - L_n - 2L_{n-2})\, b_0 + (F_n - L_n + (-1)^n)\, b_1 \\
&\quad + (F_{n-1} - L_{n-1} - (-1)^n)\, c_0 + (F_n - L_{n-1} - L_{n-2})\, c_1, \\
3b_n &= (F_{n-1} - L_n - 2L_{n-2})\, a_0 + (F_n - L_{n-1} - L_{n-2})\, a_1 \\
&\quad + (F_{n-1} + 2L_n + L_{n-2})\, b_0 + (F_n + 2L_{n-1} - L_{n-2})\, b_1 \\
&\quad + (F_{n-1} - L_n + L_{n-2})\, c_0 + (F_n - L_{n-1} + 2L_{n-2})\, c_1, \\
3c_n &= (F_{n-1} - L_{n-1} - (-1)^n)\, a_0 + (F_n - L_n + (-1)^n)\, a_1 \\
&\quad + (F_{n-1} - L_n + L_{n-2})\, b_0 + (F_n + L_n - 2L_{n-1} + L_{n-2} - (-1)^n)\, b_1 \\
&\quad + (F_{n-1} + L_n + L_{n-1} - L_{n-2} + (-1)^n)\, c_0 + (F_n + 2L_{n-1} - L_{n-2})\, c_1.
\end{aligned}$$

3. THE EXPLICIT FORMULAS

We give explicit formulas for the various recurrences we have introduced in terms of certain numbers that we are about to define.

Let

$$R = \left(\sqrt{\sqrt{13}+5} + \sqrt{\sqrt{13}-3}\right)/\sqrt{8}, r = \left(\sqrt{\sqrt{13}+5} - \sqrt{\sqrt{13}-3}\right)/\sqrt{8},$$

$$\theta = \cos^{-1}\left(-\sqrt{\frac{3+\sqrt{13}}{8}}\right), \quad \phi = \cos^{-1}\left(\sqrt{\frac{3+\sqrt{13}}{8}}\right),$$

$$S = \frac{1}{3} + \sqrt[3]{\frac{29+\sqrt{837}}{54}} + \sqrt[3]{\frac{29-\sqrt{837}}{54}},$$

$$\alpha = \cos^{-1}\frac{1}{2S\sqrt{S}}, \quad \beta = \cos^{-1}\left(-\frac{1}{2S\sqrt{S}}\right).$$

Then

$$\begin{aligned} 39H_n &= 10R^{n+3}\cos(n+3)\theta - 14R^{n+2}\cos(n+2)\theta - 18R^{n+1}\cos(n+1)\theta + 28R^n\cos n\theta \\ &\quad + 10r^{n+3}\cos(n+3)\phi - 14r^{n+2}\cos(n+2)\phi - 18r^{n+1}\cos(n+1)\phi + 28r^n\cos n\phi, \end{aligned}$$

$$\begin{aligned} G_{3n} &= H_{3n+2} - H_{3n}, \quad G_{3n+1} = H_{3n+2}, \quad G_{3n+2} = H_{3n} - H_{3n+1}, \\ J_{3n} &= H_{3n+1} + H_{3n+2}, \quad J_{3n+1} = -H_{3n}, \quad J_{3n+2} = H_{3n} - H_{3n+2}, \end{aligned}$$

$$31K_n = (6S^2 - 2S + 9)(-S)^n + 2\left(\frac{6}{S}\cos(n+2)\alpha + \frac{2}{\sqrt{S}}\cos(n+1)\alpha + 9\cos n\alpha\right)/\sqrt{S}^n,$$

$$31L_n = 2(-2S\cos(n+2)\beta - 3\sqrt{S}\cos(n+1)\beta + 9\cos n\beta)\sqrt{S}^n + \left(-\frac{2}{S^2} - \frac{3}{S} + 9\right)/S^n$$

and

$$2M_n = \cos\frac{n\pi}{4} + \cos\frac{3n\pi}{4}.$$

4. THE DERIVATIONS

We start with system (1),

$$\begin{aligned} a_{n+2} &= a_{n+1} + b_n, \\ b_{n+2} &= b_{n+1} + c_n, \\ c_{n+2} &= c_{n+1} + a_n. \end{aligned}$$

Let $A(t) = \sum_{n\geq 0} a_n t^n,\ B(t) = \sum_{n\geq 0} b_n t^n,\ C(t) = \sum_{n\geq 0} c_n t^n$. Then

$$A(t) = tA(t) + t^2B(t) + a_0 + (a_1 - a_0)t,$$

$$B(t) = tB(t) + t^2C(t) + b_0 + (b_1 - b_0)t,$$

$$C(t) = tC(t) + t^2A(t) + c_0 + (c_1 - c_0)t.$$

That is,

$$\begin{pmatrix} 1-t & -t^2 & 0 \\ 0 & 1-t & -t^2 \\ -t^2 & 0 & 1-t \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} a_0 + (a_1 - a_0)t \\ b_0 + (b_1 - b_0)t \\ c_0 + (c_1 - c_0)t \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \frac{1}{1-3t+3t^2-t^3-t^6} \begin{pmatrix} (1-t)^2 & t^2(1-t) & t^4 \\ t^4 & (1-t)^2 & t^2(1-t) \\ t^2(1-t) & t^4 & (1-t)^2 \end{pmatrix} \begin{pmatrix} a_0 + (a_1 - a_0)t \\ b_0 + (b_1 - b_0)t \\ c_0 + (c_1 - c_0)t \end{pmatrix}.$$

In particular,

$$A(t) =$$

$$\frac{a_0(1-3t+3t^2-t^3) + a_1(t-2t^2+t^3) + b_0(t^2-2t^3+t^4) + b_1(t^3-t^4) + c_0(t^4-t^5) + c_1t^5}{(1-t-t^2)(1-2t+2t^2-t^3-t^4)}.$$

Partial fractions then gives

$$\begin{aligned}
A(t) = a_0 &\left(\frac{\frac{1}{3} - \frac{1}{3}t}{1-t-t^2} + \frac{\frac{2}{3} - \frac{4}{3}t + t^2 - \frac{1}{3}t^3}{1-2t+2t^2-t^3+t^4} \right) \\
+ a_1 &\left(\frac{\frac{1}{3}t}{1-t-t^2} + \frac{\frac{2}{3}t - \frac{2}{3}t^2 + \frac{1}{3}t^3}{1-2t+2t^2-t^3+t^4} \right) \\
+ b_0 &\left(\frac{\frac{1}{3} - \frac{1}{3}t}{1-t-t^2} + \frac{-\frac{1}{3} + \frac{2}{3}t - \frac{1}{3}t^3}{1-2t+2t^2-t^3+t^4} \right) \\
+ b_1 &\left(\frac{\frac{1}{3}t}{1-t-t^2} + \frac{-\frac{1}{3}t + \frac{1}{3}t^2 + \frac{1}{3}t^3}{1-2t+2t^2-t^3+t^4} \right) \\
+ c_0 &\left(\frac{\frac{1}{3} - \frac{1}{3}t}{1-t-t^2} + \frac{-\frac{1}{3} + \frac{2}{3}t - t^2 + \frac{2}{3}t^3}{1-2t+2t^2-t^3+t^4} \right) \\
+ c_1 &\left(\frac{\frac{1}{3}t}{1-t-t^2} + \frac{-\frac{1}{3}t + \frac{1}{3}t^2 - \frac{2}{3}t^3}{1-2t+2t^2-t^3+t^4} \right).
\end{aligned}$$

If we now define the G_n by

$$\sum_{n\geq 0} G_n t^n = \frac{1}{1-2t+2t^2-t^3+t^4}$$

it follows that

$$\begin{aligned}3a_n &= (F_{n-1}+2G_n-4G_{n-1}+3G_{n-2}-G_{n-3})\,a_0+(F_n+2G_{n-1}-2G_{n-2}+G_{n-3})\,a_1\\ &\quad+(F_{n-1}-G_n+2G_{n-1}-G_{n-3})\,b_0+(F_n-G_{n-1}+G_{n-2}+G_{n-3})\,b_1\\ &\quad+(F_{n-1}-G_n+2G_{n-1}-3G_{n-2}+2G_{n-3})\,c_0+(F_n-G_{n-1}+G_{n-2}-2G_{n-3})\,c_1,\end{aligned}$$

as stated. The formulas for b_n and c_n in system (1) can be found in similar fashion.

In the case of system (2), the characteristic polynomial (the determinant of the matrix) is

$$1-3t^2-t^3+3t^4-t^6=(1-t-t^2)(1+t-t^2-t^3+t^4)$$

and if we define the H_n by

$$\sum_{n\geq 0} H_n t^n = \frac{1}{1+t-t^2-t^3+t^4}$$

then the stated results follow.

In the case of system (3) the characteristic polynomial is

$$1-t^3-3t^4-3t^5-t^6=(1-t-t^2)(1+t+2t^2+2t^3+t^4)$$

and if we define the J_n by

$$\sum_{n\geq 0} J_n t^n = \frac{1}{1+t+2t^2+2t^3+t^4}$$

then the stated results follow.

In the case of system (4), the characteristic polynomial is

$$1-4t^3-t^6=(1-t-t^2)(1-\omega t-\omega^2t^2)(1-\overline{\omega}t-\overline{\omega}^2t^2)$$

where ω, $\overline{\omega}$ are cube roots of unity, and it is not necessary to define any subsidiary sequence.

In the case of system (5), the characteristic polynomial is

$$1-t-2t^2+2t^3-t^4+t^6=(1-t-t^2)(1-t)(1+t+t^3)$$

and if we define the K_n by

$$\sum_{n\geq 0} K_n t^n = \frac{1}{1+t+t^3}$$

the stated results follow.

In the case of system (6), the characteristic polynomial is

$$1 - t - t^2 + t^4 - t^5 - t^6 = (1 - t - t^2)(1 + t^4)$$

and if we define the M_n by

$$\sum_{n \geq 0} M_n t^n = \frac{1}{1 + t^4}$$

the stated results follow.

In the case of system (7), the characteristic polynomial is

$$1 - t^2 - 2t^3 - 2t^4 + t^5 + t^6 = (1 - t - t^2)(1 + t)(1 + t^2 - t^3)$$

and if we define the L_n by

$$\sum_{n \geq 0} L_n t^n = \frac{1}{1 + t^2 - t^3}$$

then the stated results follow.

Now we turn to the proofs of the explicit formulas for $G_n,\ H_n,\ J_n,\ K_n,\ L_n$ and M_n.

First let us dispense with the M_n. We have

$$\sum_{n \geq 0} M_n t^n = \frac{1}{1 + t^4} = 1 - t^4 + t^8 - \cdots .$$

It follows that $\{M_n\} = \{1, 0, 0, 0, -1, 0, 0, 0, 1, 0, \cdots\}$, and it is easy to check that the stated formula behaves in precisely the same manner.

To deal with the K_n and L_n, we begin with the factorizations of the relevant cubics,

$$1 + t + t^3 = (1 + St)\left(1 - \frac{2}{\sqrt{S}}\cos\alpha t + \frac{1}{S}t^2\right) = (1 + St)\left(1 - \frac{1}{\sqrt{S}}e^{i\alpha}t\right)\left(1 - \frac{1}{\sqrt{S}}e^{-i\alpha}t\right)$$

and

$$1 + t^2 - t^3 = \left(1 - \frac{1}{\sqrt{S}}t\right)\left(1 - 2\sqrt{S}\cos\beta t + St^2\right) = \left(1 - \frac{1}{S}t\right)(1 - \sqrt{S}e^{i\beta}t)\left(1 - \sqrt{S}e^{-i\beta}t\right),$$

both of which are easily verified. Use of partial fractions then yields the stated formulas. (Factorization of the cubics is fairly standard – I used Cardano's method.)

Finally we come to the $G_n,\ H_n$ and J_n.

Observe that the product of the three denominators is, most remarkably, a function of t^3.

$$(1 - 2t + 2t^2 - t^3 + t^4)(1 + t - t^2 - t^3 + t^4)(1 + t + 2t^2 + 2t^3 + t^4) = 1 + t^3 + 5t^6 - t^9 + t^{12}.$$

Thus we have

$$\sum_{n\geq 0} G_n t^n = \frac{1}{1-2t+2t^2-t^3+t^4} = \frac{1+2t+2t^2+2t^3+t^4-2t^5-t^6+t^7+t^8}{1+t^3+5t^6-t^9+t^{12}},$$

$$\sum_{n\geq 0} H_n t^n = \frac{1}{1+t-t^2-t^3+t^4} = \frac{1-t+2t^2-t^3+t^4+t^5+2t^6+t^7+t^8}{1+t^3+5t^6-t^9+t^{12}}$$

$$\sum_{n\geq 0} J_n t^n = \frac{1}{1+t+2t^2+2t^3+t^4} = \frac{1-t-t^2+2t^3+t^4-2t^5+2t^6-2t^7+t^8}{1+t^3+5t^6-t^9+t^{12}}.$$

It follows that

$$\sum_{n\geq 0} G_{3n} t^n = \frac{1+2t-t^2}{1+t+5t^2-t^3+t^4}, \quad \sum_{n\geq 0} G_{3n+1} t^n = \frac{2+t+t^2}{1+t+5t^2-t^3+t^4},$$

$$\sum_{n\geq 0} G_{3n+2} t^n = \frac{2-2t-t^2}{1+t+5t^2-t^3+t^4}, \quad \sum_{n\geq 0} H_{3n} t^n = \frac{1-t+2t^2}{1+t+5t^2-t^3+t^4},$$

$$\sum_{n\geq 0} H_{3n+1} t^n = \frac{-1+t+t^2}{1+t+5t^2-t^3+t^4}, \quad \sum_{n\geq 0} H_{3n+2} t^n = \frac{2+t+t^2}{1+t+5t^2-t^3+t^4},$$

$$\sum_{n\geq 0} J_{3n} t^n = \frac{1+2t+2t^2}{1+t+5t^2-t^3+t^4}, \quad \sum_{n\geq 0} J_{3n+1} t^n = \frac{-1+t-2t^2}{1+t+5t^2-t^3+t^4},$$

$$\sum_{n\geq 0} J_{3n+2} t^n = \frac{-1-2t+t^2}{1+t+5t^2-t^3+t^4},$$

from which we can read off the stated relations between the G_n, H_n and J_n.

Indeed, we see that the G_n, H_n and J_n can all be written in terms of another sequence, the P_n, defined by

$$\sum_{n\geq 0} P_n t^n = \frac{1}{1+t+5t^2-t^3+t^4}.$$

However, we have chosen to give the G_n and J_n in terms of the H_n, simply because the characteristic polynomial of the H_n has a neat factorization,

$$\begin{aligned}1+t-t^2-t^3+t^4 &= (1-2R\cos\theta t+R^2t^2)(1-2r\cos\phi t+r^2t^2)\\ &= (1-Re^{i\theta}t)(1-Re^{-i\theta}t)(1-re^{i\phi}t)(1-re^{-i\phi}t)\end{aligned}$$

where $R,\ r,\theta$ and ϕ are given above.

The expression for the H_n follows via partial fractions.

One final comment: Factorization of the quartic was quite a challenge for me. I began with the observation that

$$1+t-t^2-t^3+t^4=(1-\omega t-t^2)(1-\overline{\omega}t-t^2),$$

where $\omega,\ \overline{\omega}$ are cube roots of unity.

In closing, I would like to acknowledge the help given me by James A. Sellers in discovering the relations between the $G_n,\ H_n$ and J_n. These discoveries greatly simplified the paper.

REFERENCES

[1] K. Atanassov. "On a Generalization of the Fibonacci Sequence in the Case of Three Sequences." *The Fibonacci Quarterly* **27** (1989): 7-10.

[2] K. Atanassov, V. Atanassova, A. Shannon and J. Turner. *New Visual Perspectives on Fibonacci Numbers.* World Scientific, 2002.

[3] K. Atanassov, J. Hlebarova and S. Mihov. "Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences." *The Fibonacci Quarterly* **30** (1992): 77-79.

[4] W. Spickerman, R. Creech and R. Joyner. "On the Structure of the Set of Difference Systems Defining the (3,F) Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **31** (1993): 333-337.

[5] W. Spickerman, R. Creech and R. Joyner. "On the (3,F) Generalizations of the Fibonacci Sequence." *The Fibonacci Quarterly* **33** (1995): 9-12.

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