Proofs that Really Count: The Art of Combinatorial Proof, by Arthur T. Benjamin and Jennifer J. Quinn. Dolciani Mathematical Exposition #27, Mathematical Association of America, 2003, 194 pp., ISBN 0-88385–333–7.

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The numbers we love—Fibonacci and Lucas numbers, binomial coefficients, primes, squares—answer questions "How many ...?" PTRC shows us the beauty and value sticking to this viewpoint for the numbers above as well as: continued fractions (numerators and denominators), linear recurrences, harmonic numbers, Stirling numbers, and more. The authors rigorously adhere to this viewpoint, developing a "proof without words" immediately perceptible demonstration of a vast collection of fundamental (and beyond) theorems about these numbers. Some of us might become exasperated: where are the building-block powerful lemmas and theorems? But their perseverance pays off in beauty laid bare. They don't recommend we eschew induction, generating functions, matrix methods, ... ; they do show us how far we can travel mathematically just exploiting counting arguments. The result is some easily grasped and remembered results (were they always thus?) and a clear appreciation of a method we probably knew but ever fully exploited.

Here is how they do it for Fibonacci numbers. Observe that there are f_n ways to tile an $n \times 1$ board (an"n-board") with squares and rectangles ($f_n = F_{n+1}$, a more convenient Fibonacci sequence for this point of view). Prove this inductively, then eschew induction for Fibonacci identities (maybe not forever, but for now). Now, consider the f_{a+b} tilings of an $a+b$-board. Either a domino covers positions a and $a+1$ or not (in the latter case we have a "fault" after position a). Immediately, we see

$$f_{a+b} = f_a f_b + f_{a-1} f_{b-1}.$$

We need never mis-remember this theorem off-by-one nor need to do the messy induction to prove it. A nice corollary of this is

$$f_{3+a} = f_3 f_a + f_2 f_{a-1} = 3f_a + 2f_{a-1} \equiv f_a \pmod 2.$$

We can get similarly easy formulas for $\sum_{k=0}^{n} f_k$ and $\sum_{k=0}^{n} f_{2k}$ (hint: in an n-tiling, where is the left most domino? where is the leftmost square?).

The Lucas number L_n is the number of *bracelets* of circumference n built with beads of size one or two (no lower case l needed here). A fault or not at the top of an n-bracelet gives us

$$L_n = f_n + f_{n-2}.$$

One final example: an n-board can be tiled using p squares and q dominoes $\binom{p+q}{p}$ different ways, which gives us

$$f_n = \sum_{n=p+2q} \binom{p+q}{p}.$$

No induction with a strange Pascal triangle diagonal, just an observation!

Nearly all of our standard repertoire of Fibonacci-Lucas formulas pop out—there are some exceptions; Benjamin and Quinn have issued a challenge in section 9.6 for the few outstanding ones.

More general linear recurrences are similarly served using tiles (or beads) of more sizes and colors. For continued fractions (like linear recurrences, but the coefficients are a function of n), instead of colors, they suggest stacking squares to some maximum height as a function of position.

Fibonacci numbers and the above-mentioned generalizations take 40% of the book (chapters 1–4 and 9). The other chapters cover

[5] Binomial identities: binomial coefficients, etc.

[6] Alternating sign binomial identities.

[7] Harmonic and Stirling number identities.

[8] Number theory: arithmetic identities, algebra, GCDs, Lucas's theorem.

After reading this book, your reviewer sees any number-theory problem and asks "what does this count?" or manipulates simple combinatorial structures and discovers old and new friends and novel representations.

This book is a treasure, with beautiful ways of seeing old, familiar mathematics and some new mathematics too. I highly recommend it.