POLYNOMIAL GENERALIZATIONS OF THE PELL SEQUENCES AND THE FIBONACCI SEQUENCE

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ABSTRACT

In this paper we present polynomial generalizations for the Pell sequence and the Fibonacci sequence together with formulas for those sequences. New combinatorial interpretations are included.

1. INTRODUCTION

In this paper, in order to find polynomial generalizations and combinatorial interpretations for the Pell sequence, we consider the identities 36 and 34 of Slater [16] that are respectively:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^5;q^8)_\infty(q^3;q^8)_\infty(q^8;q^8)_\infty \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^7;q^8)_\infty(q^1;q^8)_\infty(q^8;q^8)_\infty \tag{1.2}$$

where

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

n a nonnegative integer.

These identities are the analytic counterparts of the Göllnitz-Gordon partition identities first found by Göllnitz [8] and then rediscovered by Gordon [7].

We start by following Andrews [2], to provide a polynomial generalization for the Pell sequence and a combinatorial interpretation for this sequence. We offer a bijection between the class of partitions that appear in the Göllnitz-Gordon identities and another class of partitions that can be obtained from the left side of (1.1). New formulas for the two related sequences are given. Details are in section 3.

A third polynomial generalization including a nice relation between the Pell numbers and the Fibonacci numbers is given in section 4.

In section 5, by making use of the Rogers-Ramanujan identities, two new combinatorial interpretations for the Fibonacci numbers are given.

2. SOME DEFINITIONS AND RESULTS

The Pell numbers 1, 2, 5, 12,... defined by $a_0 = 1; a_1 = 2; a_n = 2a_{n-1} + a_{n-2}$ are the denominators of the sequence of rational numbers:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots \tag{2.1}$$

that are the continued fraction convergent to $\sqrt{2}$.

The Gaussian polynomials are defined as follows:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{if} \quad 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

For more details see Andrews [1].

When dealing with the expression

$$(1 + x + x^2)^n \tag{2.3}$$

we call the coefficients of x^j in the expanded form of (2.3) the trinomial coefficients.

It is easy to show that if

$$(1 + x + x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{2.4}$$

then

$$\binom{n}{j}_2 = \sum_{h \ge 0} (-1)^h \binom{n}{h} \binom{2n - 2h}{n - j - h} \tag{2.5}$$

The following expression (Andrews & Baxter [6]) is a q-analog of the trinomial coefficient in the same way that the Gaussian polynomial is a q-analog of the binomial coefficient, that is, its limit, when q approaches 1, is equal to the trinomial coefficient given by (2.6).

$$T_0(m, A, q) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m - 2j \\ m - A - j \end{bmatrix} \tag{2.6}$$

In order to condense the notation the following expression is defined

$$U(m, A, q) = T_0(m, A, q) + T_0(m, A + 1, q) \tag{2.7}$$

3. POLYNOMIAL GENERALIZATIONS AND COMBINATORIAL INTERPRETATIONS FOR THE PELL SEQUENCE

3.1 A first polynomial generalization for the Pell sequence

We mentioned that, following Andrews [2] one can introduce a parameter t in the left hand side of (1.1),

$$f(q, t) = \sum_{n=0}^{\infty} \frac{(-tq; q^2)_n t^n q^{n^2}}{(t; q^2)_{n+1}} \tag{3.1}$$

From here a functional equation can be obtained:

$$(1 - t) f(q, t) = 1 + (1 + tq) tq f(q, tq^2). \tag{3.2}$$

Knowing that the coefficient of t^n in the expansion of (3.1) is a polynomial in q, i.e., that

$$f(q,t) = \sum_{n=0}^{\infty} P_n(q)t^n \tag{3.3}$$

it is easy to see that

$$\begin{aligned} &P_0(q) = 1 \\ &P_1(q) = 1 + q \qquad \text{and} \\ &P_n(q) = (1 + q^{2n-1})P_{n-1}(q) + q^{2n-2}P_{n-2} \end{aligned} \tag{3.4}$$

The whole procedure described above can be easily done with A. Sills' RRtools Maple package [15].

The family of polynomials (3.4) appears in Gordon [7]. $P_m(q)$ is interpreted as a generating function for partitions of the form $n = n_1 + n_2 \ldots + n_k$ where $n_1 \leq 2m - 1$, $n_i \geq n_{i+1} + 2$ and $n_i \geq n_{i+1} + 3$ if n_i is even.

Here we concentrate on the alternative combinatorial interpretation. The equation (3.1) written in the following form:

$$f(q,t) = \frac{1}{1-t} \sum_{n=0}^{\infty} \frac{(1+tq)(1+tq^3)\ldots(1+tq^{2n-1})t^n q^{1+3+5+\cdots+2n-1}}{(1-tq^2)(1-tq^4)\ldots(1-tq^{2n})}$$

tells us that in this sum the coefficient of $t^N q^M$ is the total number of partitions of M into exactly N parts in which every odd less than or equal to the largest part appears at least once and at most twice.

By taking into consideration the factor $\frac{1}{(1-t)}$ we have proved the following theorem:

Theorem 3.1: $P_n(q)$ is the generating function for partitions into at most n parts in which every odd less than or equal to the largest part appears at least once and at most twice.

To see what is the relation between this family of polynomials and the Pell sequence we may replace q by 1 in (3.4). By doing this we get

$$\begin{aligned} &P_0(1) = 1 \\ &P_1(1) = 2 \\ &P_n(1) = 2P_{n-1}(1) + P_{n-2}(1) \end{aligned} \tag{3.5}$$

which is the Pell sequence given at the beginning of section 2.

¿From this observation we get the following combinatorial interpretation for the Pell sequence which we state as a theorem.

Corollary 3.2: The total number of partitions into at most n parts in which every odd less that or equal to the largest part appears at least once and at most twice is equal to the Pell number $P_n(1)$.

In [5] Andrews proved the following explicit formula for the family of polynomials given by (3.4):

$$P_n(q) = \sum_{j=-\infty}^{\infty} q^{16j^2+2j} U(n, 8j) - \sum_{j=-\infty}^{\infty} q^{16j^2-14j+3} U(n, 3-8j) \tag{3.6}$$

where $U(n,A) = U(n,A,q)$ given by (2.7).

Knowing that $P_n(1)$ is the Pell sequence, as we have seen in (3.5), we can get an explicit formula for this sequence by making use of formula (3.6)

$$\begin{aligned}
P_n(1) &= \lim_{q\to 1} P_n(q) = \lim_{q\to 1} \sum_{j=-\infty}^{\infty} q^{16j^2+2j} U(n,8j) - \sum_{j=-\infty}^{\infty} q^{16j^2-14j+3} U(n,3-8j) \\
&= \sum_{j=-\infty}^{\infty} \lim_{q\to 1} (T_0(n,8j,q) + T_0(n,8j+1,q) - T_0(n,3-8j,q) - T_0(n,4-8j,q)) \\
&= \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j}_2 + \binom{n}{8j+1}_2 - \binom{n}{3-8j}_2 - \binom{n}{4-8j}_2 \right] \\
&= \sum_{j=-\infty}^{\infty} \left[\binom{n+1}{8j+1}_2 - \binom{n+1}{8j+3}_2 \right]
\end{aligned}$$

It is natural to look for a bijection between the class of partitions defined by Gordon and the class appearing in theorem 3.1. It is sufficient to define a bijection that takes a partition of n where the biggest part is $2m-1$ or $2m-2$, and satisfies the conditions defined by Gordon into a partition of n in exactly m parts in which every odd less then or equal to the largest part appears at least once and at most twice. We give one as follows:

Take a partition from the class defined by Gordon. Let the number of parts of that partition be k. Let the biggest part be $2m-2$ or $2m-1$. Fix the number $2k-1$ from the biggest part, $2k-3$ from the second biggest part and proceed in the same manner until fixing one from the smallest part. Note that this is possible since $n_i \geq n_{i+1}+2$ and that, for this reason and $n_i \geq n_{i+1}$ if n_i is even, if nothing or just one is left out from the biggest part, then nothing is left out from the second biggest part and similarly for the other parts. If one is left out we take it as a new part. See the illustration below.

If the biggest part is $\geq 2k+1$ take two from the part of it that was not fixed, two from the second biggest part, and so on, until there is a part from which only one (or nothing) can be taken. If there is one, we take it. From the "taken" twos and possible one we make a new part for the new partition being formed. If there are still some parts that are left out we form another part in the same way. The partition obtained satisfies the conditions given in the theorem 3.1. We observe the following:

1. A partition with m parts is obtained. To see this note that the number left out after fixing $2k-1$ in the biggest part is minimally $2(m-k)-1$, so we are actually adding $m-k$ parts.

2. The biggest new-formed part is smaller then or equal to $2k$-so there is no need to add new odd parts.

3. By construction the new parts created form a non-increasing sequence, odd number can be formed only once -forming it twice would mean taking two ones from the same original part, which is impossible by construction. This argument also proves that the mapping defined by this procedure is injective.

The inversion mapping is defined similarly: Take a partition defined by conditions in the theorem 3.1. that has m parts. We let one copy of all odd parts fixed. Let the biggest of the fixed parts be $2k-1$. Now, each of the remaining parts is transformed in the following manner: The biggest remaining part is divided in a way that two is added to the biggest fixed part, two to the second biggest part, etc. If the number divided is odd we add one to some fixed part at the end. Note that according to the partition definition the biggest remaining part cannot be bigger than $2k$, so it can be divided among the fixed parts. The second biggest part (and all remaining) is divided in the same manner, always starting by adding two's to the biggest fixed part.

It is obvious that the resulting partition satisfies Gordon's conditions. Its biggest part is, by construction, smaller than or equal to $2k-1+2(m-k)$ and bigger than or equal to $2k-1+2(m-k)-1=2m-2$, the last -1 on the left side corresponding to the case when $m-k=1$ and the collected part is one.

3.2 A second polynomial generalization for the Pell sequence

By considering, now, a two variable function $f_{34}(q,t)$ associated with equation 34 of Slater [16] given in (1.2)

$$f_{34}(q,t)=\sum_{n=0}^{\infty}\frac{(-tq;q^2)_n t^n q^{n^2+2n}}{(t;q^2)_{n+1}} \tag{3.7}$$

and following the same steps used to get (3.2) we may obtain the functional equation

$$(1-t)f_{34}(q,t)=1+(1+tq)tq^3 f_{34}(q,tq^2) \tag{3.8}$$

and by replacing $f_{34}(q,t)$ with

$$\sum_{n=0}^{\infty} D_n(q)t^n$$

in (3.7) we can get

$$\begin{aligned} &D_0(q)=1;\\ &D_1(q)=1+q^3\\ &D_n(q)=(1+q^{2n+1})D_{n-1}(q)+q^{2n}D_{n-2}(q) \end{aligned} \tag{3.9}$$

To find a combinatorial interpretation for this family of polynomials we can write (3.7) in the following form:

$$\sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n^2+2n}}{(t;q^2)_{n+1}} =$$

$$\frac{1}{(1-t)} \sum_{n=0}^{\infty} \frac{(1+tq)(1+tq^3)\dots(1+tq^{2n-1})t^n q^{(2+1)+(4+1)\dots(2n-2+1)+(2n+1)}}{(1-tq^2)(q-tq^4)\dots(1-tq^{2n})}$$

which tells us that in this sum the coefficient of $t^N q^M$ is the total number of partitions of M into exactly N parts in which the largest part is odd, appearing only once, and every odd smaller than the largest part and greater than or equal to 3 appears at least once and at most twice.

By considering the factor $1/(1-t)$ and that for $q=1$

$$\begin{aligned} D_0(1) &= 1 \\ D_1(1) &= 2 \\ D_n(1) &= 2D_{n-1}(1) + D_{n-2}(1) \end{aligned} \tag{3.10}$$

which is the Pell sequence, we have proved the following:

Theorem 3.3: The total number of partitions into at most n parts in which the largest part is odd, greater than or equal to 3, appearing only once, and every odd smaller than the largest part and greater than or equal to 3 appears at least once and at most twice is equal to the Pell number $D_n(1)$.

For the family of polynomials (3.9) an explicit formula in terms of the q-trinomial coefficients can be also found in Andrews [5]:

$$D_n(q) = \sum_{j=-\infty}^{\infty} q^{16j^2+6j} U(n, 8j+1) - \sum_{j=-\infty}^{\infty} q^{16j^2-10j+1} U(n, -8j+2). \tag{3.11}$$

From which we have, again, the following formula for the Pell numbers:

$$\sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+1}_2 + \binom{n}{8j+2}_2 - \binom{n}{-8j+2}_2 - \binom{n}{-8j+3}_2 \right]$$

$$= \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+1}_2 - \binom{n}{8j+3}_2 \right] \tag{3.12}$$

This formula was proved in [14] by making use of an identity of Lebesgue that is also included in Slater's list.

It is an easy matter to give a bijection between the two distinct interpretations for the Pell sequence given in the corollary 3.2 and the theorem 3.3.

In the class of partitions described in the theorem 3.3 every odd part larger than or equal to 3 appears at least once and at most twice. We can subtract 2 from just one copy of each of those odd parts. By doing this we obtain a partition in the class described in theorem 3.2.

It is clear that this procedure can be inverted. Note that now the largest part obtained is necessarily odd.

We illustrate this in the table 2 where in the first column we have the partitions as described by the corollary 3.2, and in the second one those described by the theorem 3.3.

Table 2

We observe that to get the partitions in column one we took $n = 3$ in (3.4) getting

$$P_3(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + q^6 + q^7 + q^8 + q^9$$

and the ones in column two by taking $n = 3$ in (3.9) getting:

$$D_3(q) = 1 + q^3 + 2q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + q^{10} + q^{11} + q^{12} + q^{15}.$$

3.3 New formulas for two related sequences

Motivated by this formula it was possible to find and prove, by induction, the following two formulas: the first one gives us the sequence S_n of the partial sums for the Pell numbers and the second the numerators N_m of the sequence of rational numbers given in 2.1.

$$S_n = \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+2}_2 - \binom{n}{8j+4}_2 \right] \tag{3.13}$$

where

$$S_1 = 1, \quad S_2 = 3, \quad S_n = 2S_{n-1} + S_{n-2} + 1; \quad n \geq 3 \tag{3.14}$$

and

$$N_n = \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j}_2 - \binom{n}{8j+4}_2 \right] \tag{3.15}$$

where

$$N_1 = 1, \quad N_2 = 3, \quad N_n = 2N_{n-1} + N_{n-2}; \quad n \geq 3. \tag{3.16}$$

4. FIBONACCI NUMBERS AND A THIRD POLYNOMIAL GENERALIZATION FOR THE PELL SEQUENCE

The following identity can be found in Göllnitz [8]:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-1})(1-q^{8n-5})(1-q^{8n-6})}. \tag{4.1}$$

Here we note that, by introducing the variable t in the following manner (again by use of Sills [15]),:

$$F(q,t) = \sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n(n+1)}}{(t;q^2)_{n+1}} \tag{4.2}$$

recurrent relation can be obtained:

$$\begin{aligned} T_0(q) &= 1 \\ T_1(q) &= 1 + q^2 \\ T_n(q) &= (1+q^{2n})T_{n-1} + q^{2n-1}T_{n-2}(q) \end{aligned}$$

where we are taking

$$F(q,t) = \sum_{n=0}^{\infty} T_n(q) t^n.$$

Therefore we have a theorem involving Pell numbers:

Theorem 4.1: The total number of partitions into at most n parts in which the largest part is even, each even smaller than the largest part appears at least once and the odd's are distinct is equal to the Pell number $T_n(1)$.

What is nice about this theorem is the fact that for the Fibonacci numbers F_n defined by $F_0 = 0; F_1 = 1$ and $F_n = F_{n-1} + F_{n-2},\ n \geq 2$ we have proved in [12] (Theorem 3.3) that:

"The total numbers of partitions into at most n parts in which the largest part is even and every even smaller than the largest part appears at least once is equal to F_{2n+1}". This tells us that by adding the restriction "distinct odd's" we move from Fibonacci with odd index F_{2n+1} to the Pell numbers $T_n(1)$.

5. FIBONACCI NUMBERS FROM THE ROGERS-RAMANUJAN IDENTITIES

The following polynomial generalization of the Fibonacci sequence has been used by Schur [17] to prove the Rogers-Ramanujan identities (see also Andrews [2[).

$$\begin{aligned} F_0(q) &= 1 \\ F_1(q) &= 1 \\ F_n(q) &= F_{n-1} + q^n F_{n-2}(q) \end{aligned} \tag{5.1}$$

which can obtained from the second of the Rogers-Ramanujan identities (Rogers [9]), that is:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})},$$

by defining the following two variable function:

$$f(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n(n+1)}}{(t;q)_{n+1}}. \tag{5.2}$$

Considering that for $q = 1$ (5.1) is the Fibonacci sequence and that (5.1) can be written in the form:

$$\sum_{n=0}^{\infty} \frac{t^{2n} q^{1+1+2+2+\ldots+n+n}}{(1-t)(tq;q)_n} \tag{5.3}$$

it is easy to see that we get a new combinatorial interpretation for the Fibonacci numbers that is stated in the following theorem:

Theorem 5.1: The total number of partitions into at most n parts in which every integer less than or equal to the largest part appears at least twice is equal to the Fibonacci number F_n.

In [10] we find, also, a two variable function similar to (5.2) related to the first Rogers-Ramanujan identity (Rogers [9]) given by:

$$f_1(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(t;q)_{n+1}} = \sum_{n=0}^{\infty} P_n(q) t^n \tag{5.4}$$

from which we get

$$(1-t) f_1(q,t) = 1 + t^2 q f_1(q,tq).$$

Knowing this functional equation one can get the following recurrence relation for $P_n(q)$:

$$\begin{aligned} P_0(q) &= 1 \\ P_1(q) &= 1 \\ P_n(q) &= P_{n-1} + q^{n-1} P_{n-2}(q). \end{aligned} \tag{5.5}$$

To get one more combinatorial interpretation for the Fibonacci numbers we observe that, for $q = 1$, (5.5) is the Fibonacci sequence and that from the first sum in (5.4) we have the following nice result:

Theorem 5.2: The total number of partitions in which the side of the Durfee squares equals to the largest part and the largest part plus the number of parts is at most n is equal to F_n.

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