FINAL DIGIT STRINGS OF POWERS WHERE THE EXPONENTS END IN 1, 3, 7 OR 9

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ABSTRACT

Given an integer $b > 1$ and a string s of decimal digits, one may ask whether there exists an integer n such that n^b (in decimal form) ends in s. This paper answers that question for the case where the exponent b is relatively prime to 10. It extends the earlier work [2], where the question was answered for cubes.

1. INTRODUCTION

Is there an integer n such that n^{314159} ends in the digits 314159? Or, how about the same question with 314159 replaced by the integer 31415962535...5275045519 formed by the first billion digits of π? In the North Central Section MAA Team Contest of November, 2000, problem 6 asked about the existence of an integer n such that n^3 ends in 2000 ones. The affirmative answer to that question sparked the investigation into the more general question: Given a string s of decimal digits, is there an integer n such that n^3 ends in s? A complete answer is given in [2], where the results fall into four cases according as the final digit of s is in $\{1,3,7,9\}$, $\{2,4,6,8\}$, $\{5\}$ or $\{0\}$. In this work, we look at the corresponding question where the exponent 3 is replaced by an arbitrary integer b ending in 1, 3, 7 or 9. (But note that the final digit string s may end in any digit.) Affirmative answers to both the opening questions above are an immediate consequence of our first theorem. It will be useful to list here the four theorems from [2] corresponding to the four above-mentioned cases.

Theorem 1a: *Let d be a positive integer and let s be a string of d decimal digits ending in 1, 3, 7 or 9. Then there is an integer n of d or fewer digits such that n^3 ends in s.*

Theorem 2a: *Let s be a string of d decimal digits ending in 2, 4, 6 or 8. Write $s = 8^p t$, with $p \geq 0$ and t not divisible by 8.*

(A) If t is odd, then there exists an integer n with n^3 ending in s.

(B) Assume that t is even.

(i) If $d \leq 3p+1$, then there is n^3 ending in s.

(ii) If $d = 3p+2$, then there is n^3 ending in s iff $t \equiv 4 \pmod 8$.

(iii) If $d \geq 3p+3$, then there is no integer n with n^3 ending in s.

Theorem 3a: *Let s be a string of d decimal digits ending in 5. Write $s = 125^p t$, with $p \geq 0$ and t not divisible by 125.*

(A) If $5 \nmid t$, then there is an integer n with n^3 ending in s.

(B) Assume that $5 | t$.

(i) If $d \leq 3p+1$, then there is n^3 ending in s.
(ii) If $d = 3p+2$, then there is n^3 ending in s iff $25|t$.
(iii) If $d \geq 3p+3$, then there is no n^3 ending in s.

Theorem 4a: *Let s be a string of decimal digits ending in 0. Then s is the final digit string of a cube if and only if the number of final zeros in s is a multiple of 3 and there is an integer m with m^3 ending in the digit string s' obtained by removing the final zeros of s.*

It appears that when one asks the corresponding question for exponents other than 3, the problem again breaks down into cases according as the final digit in the *exponent* is in $\{1,3,7,9\}$, $\{2,4,6,8\}$, $\{5\}$ or $\{0\}$. In the final section of [2], several projects are proposed, one of which is to obtain the counterparts of the above theorems for powers higher than 3, with the exponents ending in 1, 3, 7 or 9. In the present paper, we carry out this project. A large part of the work in this paper was contained in the first author's thesis [1] in partial fulfillment of the requirements for the Bachelor of Arts with Honors in Mathematics at Concordia College, under the direction of the second author. The theorems parallel very closely the four theorems quoted above, Theorem N here corresponding to Theorem Na above.

2. STRINGS ENDING IN 1, 3, 7 OR 9

The strategy for Theorem 1 is the same as that in the proof of Theorem 1a; we proceed by induction on the length d of the string s, and in the induction step we add a digit c to the front of the integer m from the previous step to build a new integer n. For a concrete example of this in the case of cubes, see Section 2 of [2]. As in [2], we use s to denote either a string of d digits or the integer represented by that string. Throughout the paper s is allowed to have initial digits equal to zero.

Theorem 1: *Let d be a positive integer and let s be a string of d decimal digits ending in 1, 3, 7 or 9. Let b be a positive integer which in decimal form ends in the digit 1, 3, 7 or 9. Then there is an integer n of d or fewer digits such that n^b ends in s.*

Proof: To begin an induction on the length d of the string, we need to show that for every b ending in 1, 3, 7 or 9 and every k in $\{1,3,7,9\}$ there is an integer n such that n^b ends in k. Obviously, 1^b always ends in 1 and 9^b ends in 9 because b is odd. Also, from the fact that $3^4 = 81$ and $3^3 = 27$, we see that 3^b ends in 3 if $b = 4p+1$ and in 7 if $b = 4p+3$. Similarly, from the fact that $7^4 = 2401$ and $7^3 = 343$, we see that 7^b ends in 7 if $b = 4p+1$ and in 3 if $b = 4p+3$. Thus the assertion is true for $d = 1$.

Now, suppose that the assertion is true for $d = r$, and consider a string

$$s = k_{r+1}, k_r, k_{r-1}, \dots, k_1,$$

where k_1 is 1, 3, 7 or 9. By the induction hypothesis, there is an integer m of r or fewer digits such that m^b ends in $b_{r+1}, k_r, k_{r-1}, \dots, k_1$, where b_{r+1} may or may not be equal to k_{r+1}. We will show that there is an integer c, $0 \leq c \leq 9$, such that $(m + c \cdot 10^r)^b$ ends in s. Note that $m + c \cdot 10^r$ is an integer of $r+1$ digits with c as its leftmost digit. Consider

$$\begin{aligned}(m + c \cdot 10^r)^b &= m^b + bm^{b-1}c \cdot 10^r + \binom{b}{2} m^{b-2}c^2 10^{2r} + \cdots + c^b 10^{br} \\ &\equiv m^b + bm^{b-1}c \cdot 10^r \pmod{10^{r+1}}.\end{aligned}$$

The last r digits of this number are $k_r, k_{r-1}, \dots, k_1$, and the digit in position $r+1$, which we want to be k_{r+1}, is $b_{r+1} + bm^{b-1}c \pmod{10}$. Because both b and m are relatively prime to 10, bm^{b-1} has an inverse mod 10. By choosing $c = (bm^{b-1})^{-1}(k_{r+1} - b_{r+1})$ we have $n = m + c \cdot 10^r$ such that n^b ends in s. The theorem then follows by induction. □

We note here, as we did in [2], the connection to elementary group theory. The strings s of length d may be regarded as the elements of $Z/(10^d)$, where Z is the ring of integers. Those strings ending in 1,3,7 or 9 constitute the multiplicative group of invertible elements of $Z/(10^d)$, and the fact that all of them occur as final digit sequences of b^{th} powers corresponds to the fact that the operation of raising to the b^{th} power is bijective on this abelian group. There are no elements of order b.

3. STRINGS ENDING IN 2, 4, 6 OR 8

Unlike the situation when s ends in 1, 3, 7 or 9, when s ends in 2, 4, 6 or 8 there may or may not be a b^{th} power ending in s. The answer depends on the number of factors of 2 in s. If that number is a multiple of b, then there is an n with n^b ending in s. For example, with $b = 7$, if $s = (128)(314\,159) = 40\,212\,352$, or any odd multiple of 128, there is a seventh power ending in s. If not, there is such an n^b if and only if the number of factors of 2 in s is at least as large as d, the number of digits in s. E.g., if $s = (256)(314\,159) = 80\,424\,704$, then s has 8 digits and 8 factors of 2, so there is a seventh power ending in s. But if $s = (256)(514\,139) = 131\,619\,584$, then s has 9 digits but only 8 factors of 2, and there is no seventh power ending in s. This is exactly what Theorem 2a states for the case $b = 3$. We begin with two lemmas.

Lemma 1: *Let s be a string of d digits ending in 2, 4, 6 or 8, and b be a positive integer ending in 1, 3, 7 or 9. If $d \geq b$ and n^b ends in s, then $2^b | s$.*

Proof: It is clear that n must be even; write $n = 2m$. Then

$$n^b = 2^b m^b = s + 10^d k$$

for some integer $k \geq 0$, and because $d \geq b$, 2^b is a divisor of $10^d k$, and therefore of s. □

The key to the proof of Theorem 2 is now given in the next lemma.

Lemma 2: *Let s be a string of d digits ending in 2, 4, 6 or 8. If $s = 2^b t$ and there is an integer m such that m^b ends in t, then there is n such that n^b ends in s. (Here if s has initial zeros as its first digits, then t is assumed to have an equal number of them.)*

Example: Suppose $s = 00\,854\,912$ and $b = 7$. Now, $854\,912 = (2^7)(6679)$, but our t will be $006\,679$. There is indeed a seventh power ending in $006\,679$, namely $39^7 = 137\,231\,006\,679$. The lemma then asserts that there is an integer n with n^7 ending in $s = 00\,854\,912$, and the proof shows how to find such an n. Our t has 6 digits, so, in the notation of the proof, $d - r = 6$, and $d = 8$; $m = 39$. For appropriate choice of c, the proof shows, $n = 2m + c10^{d-r} = 78 + c10^6$ will work. By following the steps in the proof, one finds that $c = 19$ works. (One needs $17 + 7c \equiv 0 \pmod{25}$.) And indeed, with $n = 19\,000\,078$, we find that n^7 is a 51-digit number ending in $00\,854\,912$.

Proof: The lemma is obviously true if $b = 1$, so we assume without loss of generality that

$$b \geq 3. \tag{3.1}$$

Let $v = \lfloor b \log_{10} 2 \rfloor$, so that $10^v < 2^b < 10^{v+1}$, and let z be the number of initial zeros in s; $0 \le z < d$. Because $10^{d-z-1} < s < 10^{d-z}$, we have

$$10^{d-z-v-2} < t = \frac{s}{2^b} < 10^{d-z-v},$$

so t has $d-r$ digits, including the z initial zeros, where

$$r = v \quad \text{or} \quad r = v+1. \tag{3.2}$$

Note that

$$v < \frac{b}{3}, \tag{3.3}$$

and

$$d - r \ge 1. \tag{3.4}$$

By hypothesis, $m^b = t + k \cdot 10^{d-r}$ for some integer k. We will show that for an appropriate choice of c, the integer $n = 2m + c \cdot 10^{d-r}$ has the property that n^b ends in s. We have

$$\begin{aligned}
n^b &= (2m + c \cdot 10^{d-r})^b \\
&= 2^b m^b + b2^{b-1}m^{b-1}c10^{d-r} + \binom{b}{2}2^{b-2}m^{b-2}c^2 10^{2(d-r)} + \cdots + c^b 10^{b(d-r)} \\
&= 2^b(t + k10^{d-r}) + b2^{b-1}m^{b-1}c10^{d-r} + \binom{b}{2}2^{b-2}m^{b-2}c^2 10^{2(d-r)} + \cdots + c^b 10^{b(d-r)} \\
&= s + 10^{d-r}(2^b k + f(c)),
\end{aligned} \tag{3.5}$$

where

$$f(c) = b2^{b-1}m^{b-1}c + \binom{b}{2}2^{b-2}m^{b-2}c^2 10^{d-r} + \cdots + c^b 10^{(b-1)(d-r)}.$$

In order that s be the final digit string of n^b, it suffices that $2^b k + f(c)$ be a multiple of 10^r, for then $n^b = s + 10^d u$ for some integer u. We first observe that every term in $2^b k + f(c)$ is a multiple of 2^r. To see this note that from (3.2), (3.3) and (3.1),

$$r + 1 \le v + 2 < \frac{b}{3} + 2 \le b,$$

so 2^b and 2^{b-1} are multiples of 2^r. In each subsequent term of $f(c)$, the exponent of 2 decreases by 1 and that of 10 increases by $d-r$, which is at least 1 by (3.4). Thus, each term has 2^r as a factor, and $2^b k + f(c)$ is a multiple of 2^r, whatever the value of c. We now show that for some choice of c, $2^b k + f(c)$ is a multiple of 5^r; i.e., $f(c) \equiv -2^b k \pmod{5^r}$ for some c.

Let $S = \{0, 1, 2, \ldots, 5^r - 1\}$ be the set of integers modulo 5^r, and consider f as a mapping of S to S. We shall show that $f\colon S \to S$ is injective (one-to-one) and therefore, because S is finite, surjective (onto). We may write

$$f(c) = a_1 c + a_2 c^2 + \cdots + a_b c^b,$$

and note that $a_1 = b2^{b-1}m^{b-1}$ cannot be a multiple of 5, for b is not, and m is not (because s ends in 2, 4, 6 or 8). On the other hand, each of $a_2, \dots, a_b$ is a multiple of 10 and therefore of 5. Suppose that $f(c_1) \equiv f(c_2) \pmod{5^r}$. Then

$$0 \equiv a_1(c_1 - c_2) + a_2(c_1^2 - c_2^2) + \cdots + a_b(c_1^b - c_2^b)$$

$$\equiv (c_1 - c_2)[a_1 + a_2(c_1 + c_2) + \cdots + a_b(c_1^{b-1} + c_1^{b-2}c_2 + \cdots + c_2^{b-1})] \pmod{5^r}.$$

The quantity in brackets is not divisible by 5 (because a_1 is not and the other a_j are), so 5^r is a factor of $c_1 - c_2$; i.e., $c_1 \equiv c_2 \pmod{5^r}$. Thus, $f(c) \equiv -2^b k \pmod{5^r}$ for some integer c, and it follows from (3.5) that

$$n^b = s + u \cdot 10^d$$

for some integer u; i.e., n^b ends in the d-digit string s. $\square$

Theorem 2: *Let d be a positive integer and s be a string of d decimal digits ending in 2, 4, 6 or 8, and let b be a positive integer relatively prime to 10. If the number of factors of 2 in s is a multiple of b, then there is an integer n with n^b ending in s. If not, then there is such an n^b if and only if the number of factors of 2 in s is at least d.*

Proof: Write $s = 2^{bp}t$, with $p \geq 0$ and $2^b \not| t$. We prove the theorem by showing:
(A) If t is odd, then there is an integer n with n^b ending in s.
(B) Assume that $2|t$.
(i) If $d \leq bp + w$ where $1 \leq w < b$ and $2^w|t$, then there is an integer n such that n^b ends in s.
(ii) If $d \geq bp + w$ where $w \geq 1$ and $2^w \not| t$, then there is no b^{th} power ending in s.

(A) is immediate from Lemma 2 and Theorem 1, because t cannot end in 5.

For (B)(i), as a first case, suppose that $d \leq bp$. Place the digit 1 in position $bp+1$ in front of s (with intermediate zeros if $d < bp$), to create a new integer s':

$$s' = 10^{bp} + s = 10^{bp} + 2^{bp}t = 2^{bp}(5^{bp} + t) = 2^{bp}t',$$

where $t' = 5^{bp} + t$ is clearly odd (because $2|t$) and does not end in 5. By part (A), there is an n with n^b ending in s' and therefore ending in s.

Now, suppose that $d = bp + w_0$ with $1 \leq w_0 \leq w \leq b - 1$. Then $s < 10^{bp+w_0}$, and we want to put a digit or string of digits c in front of s to form the string s'. Let

$$s' = c \cdot 10^{bp+w_0} + s = 2^{bp}(c \cdot 2^{w_0}5^{bp+w_0} + t).$$

By hypothesis, $2^w|t$, so $2^{w_0}|t$. Write $t = 2^{w_0}t_0$, and we have

$$s' = 2^{bp}(c \cdot 2^{w_0}5^{bp+w_0} + 2^{w_0}t_0) = 2^{bp+w_0}(c \cdot 5^{bp+w_0} + t_0).$$

Because 5^{bp+w_0} is relatively prime to 2^{bk-w_0} for $k \geq 1$, we may choose an integer c_k in $\{0, 1, \dots, 2^{bk-w_0} - 1\}$ to make $(c_k 5^{bp+w_0} + t_0)$ a multiple of 2^{bk-w_0}, and for such c_k,

$$s' = c_k \cdot 10^{bp+w_0} + s = 2^{bp+w_0}2^{bk-w_0}u = 2^{b(p+k)}u = 2^{bq}t'$$

for some integer u, where $q \geq p + k$ and $2^b \not| t'$, and we have allowed for the possibility that u itself has factors of 2^b. ¿From the fact that

$$c_k < 2^{bk-w_0} < 2^{bk+b-1} < 10^{\lceil \frac{bk+b-1}{3} \rceil},$$

we see that

$$s' < 10^{bp+w_0} 10^{\lceil \frac{bk+b-1}{3} \rceil},$$

and for large enough k, the number d' of digits in s' satisfies

$$d' \leq bp + w_0 + \left\lceil \frac{bk+b-1}{3} \right\rceil \leq bp + bk \leq bq.$$

For such k, we have bq factors of 2 in s' and at most bq digits in s', so by the first case above, if t' is even, or by (A) if t' is odd, there is an integer n with n^b ending in s' and therefore ending in s.

We now prove (B)(ii). As a first case, suppose that $d = bp + w$, with $w \geq b$. If $p = 0$, the assertion follows from Lemma 1. Thus, assume that $p \geq 1$, and suppose, on the contrary, that there is an integer n with n^b ending in s:

$$n^b = s + k10^d = 2^{bp}t + k10^{bp+w} = 2^{bp}(t + k2^w 5^{bp+w}).$$

Then $2^{bp}|n^b$, so $2^p|n$. Write $n = 2^p m$. Then

$$2^{bp} m^b = n^b = 2^{bp}(t + k2^w 5^{bp+w}),$$

and

$$m^b = t + k2^w 5^{bp+w} = t + k5^{bp}10^w.$$

Because $w \geq b$, m^b ends in the last b digits of t. Now,

$$t = \frac{s}{2^{bp}} > \frac{s}{10^{\lceil \frac{bp}{3} \rceil}},$$

so t has at least $bp + w - \lceil \frac{bp}{3} \rceil > w \geq b$ digits, while $2^b \not| t$, contradicting Lemma 1. Hence, there is no n with n^b ending in s.

In the remaining case we have $d = bp + w$ and $1 \leq w \leq b - 1$. Suppose that there is n with n^b ending in s:

$$n^b = s + k10^d = 2^{bp}t + k10^{bp+w} = 2^{bp}(t + k2^w 5^{bp+w}).$$

This shows that $2^{bp}|n^b$, so $2^p|n$; write $n = 2^p m$. Then

$$m^b = t + k2^w 5^{bp+w}.$$

Recall that $2|t$, so $2|m^b$ and therefore $2|m$ and $2^b|m^b$. But with $w \leq b - 1$, this implies that $2^w|m^b$ and hence that $2^w|t$, a contradiction. This completes the proof of Theorem 2. $\square$

4. STRINGS ENDING IN 5

In this section, we deal with the case where s ends in 5. There is a close parallel with Theorem 2, where the number of factors of 2 in s determines whether or not there is a b^{th} power ending in s. Here it is the number of factors of 5. Although the arguments are much the same as in the preceding section, the lemmas necessary to obtain Theorem 3 are a bit more complex than those for Theorem 2. Lemma 3 below is preparatory to Lemma 4, which says that for $d \leq b+1$, there is n^b ending in s if and only if s (or the last b digits of s, in case $d = b+1$) occurs as the final digit string of an odd multiple of 5^b less than 10^b.

Lemma 3: *Let q and r be odd positive integers. Then $(5q)^b$ and $(5r)^b$ have the same final b-digit sequences if, and only if, q and r differ by a multiple of 2^b.*

Proof: Assume that $(5q)^b$ and $(5r)^b$ have the same final b-digit sequences. Then

$$(5q)^b - (5r)^b = 10^b a$$

for some integer a, whence

$$5^b(q^b - r^b) = 10^b a = 2^b 5^b a,$$

so that

$$2^b a = q^b - r^b = (q-r)(q^{b-1} + q^{b-2}r + \cdots + qr^{b-2} + r^{b-1}).$$

Each term in the last factor on the right is odd because both q and r are odd. Furthermore, the number of terms, b, is odd, so that factor is odd. It follows that $q - r$ is a multiple of 2^b.

Conversely, suppose that $q - r$ is a multiple of 2^b. Then $q = r + 2^b k$ for some integer k, and

$$q^b = (r + 2^b k)^b = r^b + 2^b u$$

for some integer u. Thus,

$$(5q)^b - (5r)^b = 5^b 2^b u = 10^b u;$$

i.e., $(5q)^b$ and $(5r)^b$ have the same final b-digit strings. □

Lemma 4: *(A) If s (ending in 5) has exactly b digits and there is a b^{th} power ending in s, then s must be one of the odd multiples of 5^b less than 10^b (including initial zeros as necessary so that the multiple has exactly b digits).*

(B) Conversely, all odd multiples of 5^b less than 10^b (again, with initial zeros if necessary) are b-digit final sequences of b^{th} powers.

(C) Furthermore, every sequence of $b+1$ digits that ends in one of the above b-digit strings is the ending digit sequence of a b^{th} power. (And, of course, these are then the only $(b+1)$-digit final sequences of b^{th} powers.)

Proof: (A) Suppose that n^b ends in the b-digit string s with final digit 5. Then $n = 5q$ for some odd integer q, and

$$(5q)^b = a10^b + s$$

for some integer a. Thus $s = 5^b q^b - a10^b$ is an odd multiple of 5^b.

(B) From Lemma 3 one sees that there are 2^{b-1} different b-digit sequences ending in 5 that are final sequences of b^{th} powers, namely the 2^{b-1} odd multiples of 5^b from $1 \cdot 5^b$ to $(2^b - 1)5^b$, supplied as necessary with initial zeros to have length b. These are all the odd multiples of 5^b which are smaller than 10^b, so (B) follows.

(C) If q and r are odd integers, then $(5q)^b$ and $(5r)^b$ have the same final $b+1$ digits if, and only if, q and r differ by a multiple of 2^{b+1}, as one shows exactly in the way that Lemma 3 was proved: If

$$(5q)^b - (5r)^b = 10^{b+1}a$$

then

$$5^b q^b - 5^b r^b = 5^{b+1}2^{b+1}a,$$

and

$$5 \cdot 2^{b+1}a = q^b - r^b = (q-r)(q^{b-1} + q^{b-2}r + \cdots + qr^{b-2} + r^{b-1}). \tag{4.1}$$

The last factor on the right above is odd, so $q-r$ is a multiple of 2^{b+1}. Equation (4.1) also shows that $5|(q^b - r^b)$, and we now show that then 5 must divide $q-r$. We show this by showing that $0^b, 1^b, 2^b, 3^b, 4^b$ are all distinct modulo 5.

Either $b = 4k+1$ or $b = 4k+3$ for some integer k. Note that by Fermat's Little Theorem (or by direct calculation) that $1^{4k}, 2^{4k}, 3^{4k}$ and 4^{4k} are all equal to 1 modulo 5. If $b = 4k+1$, then $0^b \equiv 0, 1^b \equiv 1, 2^b \equiv 2, 3^b \equiv 3$ and $4^b \equiv 4$ modulo 5. If $b = 4k+3$, then $0^b \equiv 0, 1^b \equiv 1, 2^b \equiv 3, 3^b \equiv 2$ and $4^b \equiv 4$ modulo 5. Hence, if $(5q)^b$ and $(5r)^b$ have the same final $(b+1)$-digit strings, then $q-r$ is a multiple of $5 \cdot 2^{b+1} = 10 \cdot 2^b$. Thus there are $10 \cdot 2^{b-1}$ different $(b+1)$-digit final strings of b^{th} powers ending in 5, and these must be all those obtained by putting an arbitrary extra digit in front of the 2^{b-1} different b-digit strings.

We omit the proof of the converse, which is done exactly as in Lemma 3. $\square$

Lemma 5: *If $d \geq b$ and n^b ends in s with final digit 5, then $5^b|s$.*

Proof: Let s_0 be the number formed by the last b digits of s. Then

$$s = s_0 + 10^b k = s_0 + 5^b 2^b k$$

for some integer k. By Lemma 4, s_0 is a multiple of 5^b, and therefore so is s. $\square$

Lemma 6: *Let $s = 5^b t$ be an integer of d digits ending in 5, with z initial zeros, where $0 \leq z < d$ and b is an integer ending in 1, 3, 7 or 9. If there is an integer m such that m^b ends in t (assumed also to have z initial zeros), then there is an integer n such that n^b ends in s.*

Proof: The case $b = 3$ is proved in [2], so we may assume that

$$b \geq 7. \tag{4.2}$$

Let $q = \lfloor b \log_{10} 5 \rfloor$, so that $10^q < 5^b < 10^{q+1}$. Because $10^{d-z-1} < s < 10^{d-z}$, we then have

$$10^{d-z-q-2} < t = \frac{s}{5^b} < 10^{d-z-q},$$

so that t has $d-r$ digits (including the initial zeros), where

$$r = q \text{ or } q+1. \tag{4.3}$$

Note that

$$q < .7b \tag{4.4}$$

and

$$d - r \geq 1. \tag{4.5}$$

By hypothesis, $m^b = t + k \cdot 10^{d-r}$ for some integer k. We shall show that for appropriate choice of the integer c, $n = 5m + c \cdot 10^{d-r}$ has the property that n^b ends in s. We have

$$\begin{aligned} n^b &= (5m + c \cdot 10^{d-r})^b \\ &= 5^b m^b + b5^{b-1}m^{b-1}c10^{d-r} + \binom{b}{2}5^{b-2}m^{b-2}c^2 10^{2(d-r)} + \cdots + c^b 10^{b(d-r)} \\ &= 5^b(t + k \cdot 10^{d-r}) + b5^{b-1}m^{b-1}c10^{d-r} + \binom{b}{2}5^{b-2}m^{b-2}c^2 10^{2(d-r)} + \cdots + c^b 10^{b(d-r)} \\ &= s + 10^{d-r}(5^b k + f(c)), \end{aligned} \tag{4.6}$$

where

$$f(c) = b5^{b-1}m^{b-1}c + \binom{b}{2}5^{b-2}m^{b-2}c^2 10^{d-r} + \cdots + c^b 10^{(b-1)(d-r)}.$$

In order that s be the final d-digit string of n^b, it suffices that $5^b k + f(c)$ be a multiple of 10^r. For starters, let's show that every term in $5^b k + f(c)$ is a multiple of 5^r. This will follow from the facts that (i) $b - 1 \geq r$, and (ii) $(d - r) \geq 1$, because of the fact that the first term in $f(c)$ has the factor 5^{b-1} (times $10^{0(d-r)}$), and in each subsequent term the exponent of 5 is reduced by 1 while that of 10 is increased by $d - r$. ¿From (4.3), (4.4), and (4.2) we have

$$r + 1 \leq q + 2 < .7b + 2 < b,$$

proving (i), and (ii) is (4.5). Thus, $5^b k + f(c)$ is a multiple of 5^r, whatever the value of c. It remains only to show that for some choice of c, $5^b k + f(c)$ is a multiple of 2^r; i.e., that $f(c) \equiv -5^b k \pmod{2^r}$ for some c.

Let $S = \{0, 1, 2, \ldots, 2^r - 1\}$ be the set of integers modulo 2^r, and consider f as a mapping from S to S. We show that this mapping is injective (one-to-one), and therefore, because S is finite, it is surjective (onto). Write

$$f(c) = a_1 c + a_2 c^2 + \cdots + a_b c^b,$$

and note that a_1 is odd (for m must be odd), but the remaining a_j are even. Suppose that $f(c_1) \equiv f(c_2) \pmod{2^r}$. Then

$$\begin{aligned} 0 &\equiv a_1(c_1 - c_2) + a_2(c_1^2 - c_2^2) + \cdots + a_b(c_1^b - c_2^b) \\ &\equiv (c_1 - c_2)[a_1 + a_2(c_1 + c_2) + \cdots + a_b(c_1^{b-1} + c_1^{b-2}c_2 + \cdots + c_2^{b-1})] \pmod{2^r}. \end{aligned}$$

The quantity in brackets is odd, so 2^r is a factor of $c_1 - c_2$; i.e., $c_1 \equiv c_2 \pmod{2^r}$. Thus, for some c, $5^b k + f(c)$ is a multiple of 2^r, and therefore of 10^r, and it now follows from (4.6) that

$$n^b = s + u \cdot 10^d$$

for some integer u; i.e., n^b ends in the d-digit string s. $\square$

Theorem 3: *Suppose that the digit string s ends in 5. If the number of factors of 5 in the integer s is a multiple of b, then there is an integer n with n^b ending in s. If not, then there is such an n^b if and only if the number of factors of 5 in s is at least as big as d, the number of digits in s.*

Remark: Although it is in fact implied by Theorem 3, it seems worth noting here again the result of Lemma 4: For $d \leq b+1$, there is an integer n with n^b ending in s if and only if s (or the last b digits of s, in case $d = b+1$) occurs as the final digit string of an odd multiple of 5^b less than 10^b.

Proof: Write $s = 5^{bp}t$ with $p \geq 0$ and $5^b \not| t$. We prove the theorem by showing:
(A) If $5 \not| t$, then there is an integer n with n^b ending in s.
(B) Assume that $5|t$.
(i) If $d \leq bp + w$, where $1 \leq w < b$ and $5^w|t$, then there is an integer n such that n^b ends in s.
(ii) If $d \geq bp + w$ where $w \geq 1$ and $5^w \not| t$, then there is no b^{th} power ending in s.

To prove (A), note first that because t is odd and not divisible by 5, t ends in 1, 3, 7 or 9. By Theorem 1, there is an integer m such that m^b ends in t. Then, by Lemma 6, there is an integer n such that n^b ends in s.

For (B)(i), first consider the case $d \leq bp$. Just as in this case in Theorem 2, let

$$s' = 10^{bp} + s = 5^{bp}(2^{bp} + t) = 5^{bp}t',$$

where $t' = 2^{bp} + t$. Note that t' is odd and not divisible by 5 because t is divisible by 5, and thus t' ends in 1, 3, 7 or 9. By Theorem 1, there is a b^{th} power ending in t', so by Lemma 6 there is a b^{th} power ending in s' and therefore ending in s.

Now suppose that $d = bp + w_0$ with $1 \leq w_0 \leq w \leq b-1$. Then $s < 10^{bp+w_0}$, and we will put a digit or string of digits c in front of s to form a new string s'. Let

$$s' = c10^{bp+w_0} + s = 5^{bp}(c5^{w_0}2^{bp+w_0} + t), \tag{4.7}$$

and note that $5^{w_0}|t$ because by hypothesis $5^w|t$. Write $t = 5^{w_0}t_0$. Then

$$s' = 5^{bp}(c5^{w_0}2^{bp+w_0} + 5^{w_0}t_0) = 5^{bp+w_0}(c2^{bp+w_0} + t_0).$$

Because 2^{bp+w_0} is relatively prime to 5^{bk-w_0} for $k \geq 1$, there is for every $k \geq 1$ an integer c_k in $\{0, 1, \ldots, 5^{bk-w_0} - 1\}$ such that $(c_k 2^{bp+w_0} + t_0)$ is a multiple of 5^{bk-w_0}. For such a c_k,

$$s' = 5^{bp+w_0}5^{bk-w_0}u = 5^{b(p+k)}u = 5^{bq}t' \tag{4.8}$$

for some integer u, where $q \geq p + k$ and $5^b \not| t'$. Here we have allowed for the possibility that u contains additonal factors of 5^b. From the first equation in (4.7) and the fact that

$$c_k < 5^{bk-w_0} < 5^{bk+b-1} < 10^{(0.7)(bk+b-1)},$$

it follows that for sufficiently large k, the number d' of digits in s' satisfies

$$d' \leq bp + w_0 + (0.7)(bk + b - 1) \leq bp + bk \leq bq. \tag{4.9}$$

For such k and the corresponding c_k, (4.8) shows that we have bq factors of 5 in s' and (4.9) shows that s' has at most bq digits. Therefore, by the first case above (if $5|t'$), or by (A) (if $5 \not| t'$), there is an integer n with n^b ending in s' and therefore ending in s.

We proceed now to (B)(ii). It suffices to prove it for $d = bp + w$. As a first case, suppose that $w \geq b$. If $p = 0$ we have $d \geq b$ but $5^b \not| s$ because $s = t$. Then Lemma 5 implies that there is no b^{th} power ending in s. Assume that $p \geq 1$. To show there is no n^b ending in s, suppose on the contrary that there is one:

$$n^b = s + k10^d = 5^{bp}t + k10^{bp+w} = 5^{bp}(t + k2^{bp}10^w).$$

Thus $5^{bp}|n^b$, so $5^p|n$ and we may write $n = 5^p m$. Then

$$5^{bp}m^b = n^b = 5^{bp}(t + k2^{bp}10^w),$$

so

$$m^b = t + k2^{bp}10^w,$$

and because $w \geq b$, m^b ends in the last b digits of t. Now,

$$t = \frac{s}{5^{bp}} > \frac{s}{10^{0.7bp}},$$

so t has at least $\lfloor bp + w - .7bp \rfloor \geq \lfloor bp + b - .7bp \rfloor \geq b$ digits. Then Lemma 5 says that $5^b|t$, contrary to our hypothesis. Thus, there is no integer n with n^b ending in s.

In the remaining case, we have $d = bp + w$ with $1 \leq w \leq b - 1$. By assumption, $5|t$ but $5^w \not| t$, so in fact $2 \leq w \leq b - 1$. Suppose that there did exist an integer n with n^b ending in s:

$$n^b = s + k10^d = 5^{bp}t + k10^{bp+w} = 5^{bp}(t + k5^w 2^{bp+w}).$$

Then $5^{bp}|n^b$, so $5^p|n$, and we may write $n = 5^p m$ for some integer m. Then

$$m^b = t + k5^w 2^{bp+w}.$$

Because $5|t$, this shows that $5|m$ and therefore $5^b|m^b$. Then $5^w|m^b$ and hence $5^w|t$, a contradiction. Thus, no such n exists. □

5. STRINGS ENDING IN 0

Having determined in the preceding sections which digit strings s, not ending in 0, are final digit strings of integer powers of the form n^b if b ends in 1, 3, 7 or 9, it is easy now to deal with strings ending in 0. If n^b ends in 0, then n itself ends in 0, and we may write $n = 10^p m$ where $10 \not| m$. Thus $n^b = 10^{bp}m^b$, and this proves Theorem 4:

Theorem 4: *Suppose that s is a string of decimal digits ending in 0 but having at least one nonzero digit. Then there exists an integer n such that n^b ends in s if and only if the number of final zeros in s is a multiple of b and there exists an integer m such that m^b ends in the string s' obtained by removing the final zeros of s. (Of course if s consists entirely of zeros there is always an integer n with n^b ending in s.)*

6. CONCLUSION

The above theorems characterise all ending digit sequences for integers of the form n^b where b is relatively prime to 10; i.e., b ends in 1, 3, 7 or 9, and thus completes Project 1 of [2]. There remain two projects suggested in [2]:

Project 2: *Investigate final digit sequences for squares of integers; for fifth powers.*

Project 3: *Investigate final digit sequences for integer powers in base 8; in base 12.*

We would suggest an expansion of Project 2, namely

Project 2A: *What are final digit sequences for numbers of the form n^b when b ends in 2, 4, 6 or 8?*

Project 2B: *What are final digit sequences for numbers of the form n^b when b ends in 5? What about exponents ending in 0?*

These projects lend themselves well to undergraduate research in that they do not appear to use deep mathematics unavailable to undergraduates, but yet are decidedly nontrivial.

REFERENCES

[1] Daniel P. Biebighauser. “Final Digit Strings of Powers Where The Exponents End in 1, 3, 7 or 9.” Senior Honors Thesis, Concordia College, 2002.

[2] Daniel P. Biebighauser, John Bullock and Gerald A. Heuer. “Final Digit Strings of Cubes.” *Mathematics Magazine* **77.2** (2004): 149-155.

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