GENERALIZED ARITHMETIC TRIANGLES VIA CONVOLUTION

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ABSTRACT

Pascal's Triangle is a convolution triangle; each polynomial that forms a diagonal can be generated by repeatedly convolving the polynomial $f(x) = 1$ with itself. We consider *Generalized Pascal Triangles*, convolution triangles whose generating polynomials are $f(x) = m$, where m is a positive integer. These generalized triangles share much in common with their progenitor. We investigate other means of generation and consider self-similarity along the same lines as C.T. Long's investigation of Pascal's Triangle in his 1981 article in *The Fibonacci Quarterly*.

1. DIAGONAL POLYNOMIALS AND CONVOLUTION

Consider the sequence of functions $f_j : \mathbb{N}^* \to \mathbb{N}$ where $\mathbb{N}^* = \mathbb{N} \cup \{0\}$:

$$\begin{aligned}
f_0(x) &= 1, \\
f_1(x) &= x + 1 = (x+1) \cdot f_0(x), \\
f_2(x) &= \frac{1}{2!}(x+1)(x+2) = \frac{1}{2}(x+2) \cdot f_1(x), \\
f_3(x) &= \frac{1}{3!}(x+1)(x+2)(x+3) = \frac{1}{3}(x+3) \cdot f_2(x), \\
&\vdots \\
f_n(x) &= \frac{1}{n!}(x+1)(x+2)\cdots(x+n) \\
&= \frac{1}{n}(x+n) \cdot f_{n-1}(x) \\
&\vdots
\end{aligned}$$

It is clear that $f_n(k) = \binom{n+k}{k}$.

It is well-known that the sum of the entries along any northeast-southwest diagonal of Pascal's Triangle down to, say, the kth entry, is equal to the entry that is southeast of the last entry summed. For example, along the 4th diagonal, where we start numbering from zero, we have

$$1 + 5 + 15 + 35 = 56.$$

This is often called the "hockey stick identity." (see Figure 1.) It tells us we can generate each diagonal of Pascal's Triangle from the diagonal that precedes it. Inductively, all diagonals can be generated from the 0th diagonal.

Figure 1: The "hockey stick" identity.

This will provide the basis for the Generalized Pascal's Triangles considered in this paper. For convolution our convention shall be to consider only diagonals that flow in the southwest direction, and to begin numbering them at zero.

The convolution of functions $g, h : \mathbb{N}^* \to \mathbb{N}^*$, is defined by

$$(g * h)(k) = \sum_{j=0}^{k} g(j) \cdot h(k-j) \quad (k \in \mathbb{N}^*).$$

Recast in the language of convolution, the hockey stick identity becomes

$$f_n(k) = f_0 * f_{n-1}(k), \tag{1}$$

so

$$f_n(k) = \underbrace{(f_0 * f_0 * \cdots * f_0)}_{n \text{ times}}(k) \tag{2}$$

for all $n \geq 0$. Since convolution is commutative and associative, a simple induction allows us to recast equations (1) and (2) as

$$f_n(k) = (f_m * f_{n-m-1})(k), \tag{3}$$

for all $n \geq 1$ and $m < n$. For example, $56 = \binom{8}{3} = f_5(3)$ can be written in any of three ways

$$\begin{aligned}
(f_0 * f_4)(3) &= 1 \cdot 35 + 1 \cdot 15 + 1 \cdot 5 + 1 \cdot 1 \\
(f_1 * f_3)(3) &= 1 \cdot 20 + 2 \cdot 10 + 3 \cdot 4 + 4 \cdot 1 \\
(f_2 * f_2)(3) &= 1 \cdot 10 + 3 \cdot 6 + 6 \cdot 3 + 10 \cdot 1
\end{aligned}$$

In terms of convolution, Pascal's Triangle can be written as shown in Figure 2.

Figure 2: Pascal's Triangle in terms of convolution.

If we choose $m \in \mathbb{N}, x \in \mathbb{N}^*, f_0(x) = m$ we may generate new integer triangles via the convolution $f_n(x) = (f_0 * f_{n-1})(x)$. Since we generate Pascal's Triangle if we choose $f_0(x) = 1$, we call these triangles defined via convolution *Generalized Pascal's Triangles* (GPTs). Using the initial constant polynomial $f_0(x) = m$ we obtain the triangle in Figure 3.

$$
\begin{array}{ccccccccccccc}
 & & & & & & m & & & & & & \\
 & & & & & m & & m^2 & & & & & \\
 & & & & m & & 2m^2 & & m^3 & & & & \\
 & & & m & & 3m^2 & & 3m^3 & & m^4 & & & \\
 & & m & & 4m^2 & & 6m^3 & & 4m^4 & & m^5 & & \\
 & m & & 5m^2 & & 10m^3 & & 10m^4 & & 5m^5 & & m^6 & \\
m & & 6m^2 & & 15m^3 & & 20m^4 & & 15m^5 & & 6m^6 & & m^7 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Figure 3: GPT generated by $f_0(x) = m$.

Though Pascal's Triangle is symmetric about a vertical axis, these GPTs are not. Thus our convention of considering diagonals which proceed downward in a southwesterly direction is necessary. By convention we define $f_{-1}(x) = 0$ and $f_n(-1) = 0$.

We call a triangle generated in this fashion, using $f_0(x) = m$, the mth Generalized Pascal's Triangle (or mth GPT). These triangles clearly have much in common with the ordinary Pascal's Triangle as do the convolution triangles of Fibonacci sequences studied by Hoggatt [3], and by Hoggatt and Bicknell [4].

2. PROPERTIES OF GPTs

Since we generate the mth GPT via convolution using $f_0(x) = m$, it is a simple task to prove by induction that the diagonal polynomials for this GPT are

$$\begin{aligned} f_n(x) &= (f_0 * f_{n-1})(x) \\ &= \frac{m}{n}(x+n) \cdot f_{n-1}(x) \\ &= \frac{m^{n+1}}{n!}(x+1)(x+2)(x+3)\cdots(x+n). \end{aligned}$$

It follows immediately that the entries in the mth GPT are

$$f_n(k) = m^{n+1}\binom{n+k}{k}. \tag{4}$$

Also the entries in the rth *row*, $r = 0, 1, 2, \ldots$, of the mth GPT are the coefficients of x in the binomial expansion of $m \cdot (x+m)^r$. Thus, the entries in the mth GPT can be derived by an addition rule similar to that for Pascal's Triangle. From (4),

$$\begin{aligned} mf_{n-1}(k) + f_n(k-1) &= m \cdot m^n \binom{n-1+k}{k} + m^{n+1}\binom{n+k-1}{k-1} \\ &= m^{n+1}\binom{n+k-1}{k} \qquad (5) \\ &= f_n(k). \end{aligned}$$

Moreover, the following generalizations, which we relate without proof, are true.

1. The entries in the nth northwest-southeast diagonal of the mth GPT are the coefficients of the MacLaurin series expansion of
$$\frac{m}{(1-mx)^{n+1}},$$
for $n \geq 0$. (For this result about Pascal's Triangle, see [2].)
2. In the mth GPT,
$$f_n(k) = \frac{m^{n+1}}{\int_0^1 (1-x^{1/k})^n dx},$$
for $n \geq 0$, $k \geq 1$. (For this result about Pascal's Triangle, see [1].)
3. The sum of the entries in the rth row of the mth GPT is $m \cdot (m+1)^r$. (In Pascal's Triangle, i.e., when $m = 1$, it is well-known that this sum is 2^r.)
4. Just as the sum of the falling diagonals of Pascal's triangle generate successive terms of the Fibonacci sequence, the sums of falling diagonals of the mth GPT generate the sequence $\{u_n\}_{n\geq 1}$ where $u_1 = u_2 = m$ and $u_{n+2} = mu_{n+1} + u_n$ for $n \geq 1$. Similarly, the sums of the rising diagonals generate the sequence $\{v_n\}_{n\geq 1}$ where $v_1 = v_2 = m$ and

$v_{n+2} = v_{n+1} + mv_n$ for $n \geq 1$. And it is well-known that these sequences possess a host of properties generalizing Fibonacci results.

3. SELF-SIMILARITY: GPTs MODULO p

C.T. Long, in his 1981 article [5], noticed that when each entry in Pascal's Triangle is replaced by its remainder modulo a prime p, some very interesting things result. We would like to follow his line of thinking for our GPTs.

Consider the mth GPT modulo a prime p. If $p|m$, every entry in the triangle is a multiple of m. Since nothing interesting occurs, we assume that $(p, m) = 1$.

The GPT (mod p) can then be partitioned into equilateral subtriangles of size p^a, where a is a positive integer so that

1. There are exactly p distinct such subtriangles.
2. Any entry not in one of these subtriangles is zero.
3. The subtriangles satisfy the "rule of addition" for the mth GPT. That is, elements of side-by-side subtriangles can be combined (pointwise, taking m times a number from the left sub-triangle and adding the corresponding member of the right subtriangle, (mod p)) to obtain the subtriangle below.
4. If we create a new triangle using the top entries from each subtriangle, we obtain the original GPT. In this way, we see we have a triangle of triangles that is "isomorphic" to the original triangle.

Figure 4: 2^{nd} GPT (mod 3) partitioned into equilateral subtriangles of size 3^1.

Figure 5: Sub-triangles can be combined pointwise using the same rule as for GPTs

We shall prove these statements via three propositions. We shall need two tools from elementary number theory. The first is Fermat's Little Theorem. The second, not so well-known, is a theorem of Lucas: If p is a prime, and a, n, k, r, and s are integers, then

$$\binom{n\cdot p^a+r}{k\cdot p^a+s}\equiv\binom{n}{k}\cdot\binom{r}{s}\pmod{p}.$$

Before we list the propositions, note that the subtriangles of size p^a have the form

We shall call each such subtriangle (for a fixed a), $\Delta_{n,k}$.

Proposition 3.1: *The topmost entry in each subtriangle equals the corresponding entry in the entire triangle. That is, for $k\geq 0$*

$$m^{p^a\cdot k+1}\cdot\binom{np^a}{kp^a}\equiv m^{k+1}\cdot\binom{n}{k}\pmod{p}.$$

This shows there are GPTs within GPTs, ad infinitum.

Proof: The result is an immediate consequence of the Fermat and Lucas Theorems. □

Proposition 3.2: $m^{p^a k+1+s}\cdot\binom{(n+1)p^a+p^a-1}{kp^a+s}\equiv 0\pmod{p}$, *for* $1<s<p^a-1$.

That is, the entries below and to the left of the bottom row of each subtriangle $\Delta_{n,k}$ are all zeros (save perhaps the leftmost one). It follows that the top entry in each $\Delta_{n,k}$ determines the entire subtriangle, and so there are exactly p distinct subtriangles $\Delta_{n,k}$ in one-to-one correspondence with the integers $0,1,2,\ldots,p-1$.

Proof: It will be enough to show that if any two entries of the bottom row of a $\Delta_{n,k}$ (except the leftmost pair) are combined using the rule of "addition" for the mth GPT, the result is 0.

By Lucas' Theorem

$$m\cdot m^{p^a k+1+s}\cdot\binom{np^a+p^a-1}{kp^a+s}+m^{p^a k+2+s}\cdot\binom{np^a+p^a-1}{kp^a+s+1}$$

$$\equiv m^{p^a k+2+s} \cdot \left[\binom{n}{k}\binom{p^a-1}{s} + \binom{n}{k}\binom{p^a-1}{s+1}\right]$$

$$\equiv m^{p^a k+2+s} \cdot \binom{n}{k}\binom{p^a}{s+1}$$

$$\equiv 0 \text{ since } s < p^a - 1 \pmod{p} \quad \square$$

Lastly, we show the subtriangles $\Delta_{n,k}$ follow the rule for addition for the mth GPT. In light of the two propositions above, we need only show that the first entries in two side-by-side subtriangles $\Delta_{n,k}$ combine to give the topmost entry in the $\Delta_{n,k}$ below them. That is, the top entries in $\Delta_{n,k}$ and $\Delta_{n,k+1}$ combine to give the top entry in $\Delta_{n+1,k+1}$.

Proposition 3.3: *Let a, k, n, and m be positive integers, and p a prime which does not divide m. Then*

$$m \cdot m^{p^a k+1}\binom{np^a}{kp^a} + m^{p^a(k+1)+1}\binom{np^a}{(k+1)p^a} \equiv m^{p^a(k+1)+1}\binom{(n+1)p^a}{(k+1)p^a}.$$

Proof: Again the Fermat and Lucas Theorems imply that

$$m \cdot m^{p^a k+1}\binom{np^a}{kp^a} + m^{p^a(k+1)+1}\binom{np^a}{(k+1)p^a}$$

$$\equiv m^{k+2}\binom{n}{k} + m^{m+2}\binom{n}{k+1}$$

$$\equiv m^{k+2}\binom{n+1}{k+1}$$

$$\equiv m^{p^a(k+1)+1}\binom{(n+1)p^a}{(k+1)p^a} \pmod{p}. \quad \square$$

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