ON IDENTITIES INVOLVING BERNOULLI AND EULER POLYNOMIALS

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ABSTRACT

A class of identities satisfied by both Bernoulli and Euler polynomials is established. Recurrence relations for Bernoulli and Euler numbers are derived.

1. INTRODUCTION

Given an even formal power series $F(t)$, we define a sequence of polynomials $\{P_m(x) : m = 0, 1, 2, \cdots\}$ by

$$e^{(x-\frac{1}{2})t}F(t) = \sum_{m=0}^{\infty} P_m(x)\frac{t^m}{m!}. \tag{1}$$

It is easy to verify that

$$\frac{d}{dx}P_{m+1}(x) = (m+1)P_m(x), \quad P_m(x+1) = (-1)^m P_m(-x); \tag{2}$$

$$P_m(x+s) = (P_0(x) + s)^m, \tag{3}$$

where in (3) and what follows in the expansion by the binomial formula, the kth power of $P_j(x)$ is, by convention, to stand for $P_{j+k}(x)$. Two important examples of $P_m(x)$ $(m = 0, 1, 2, \cdots)$ are Bernoulli polynomials $B_m(x)$ and Euler polynomials $E_m(x)$, for which

$$F(t) = \frac{te^{\frac{1}{2}t}}{e^t - 1} \quad \text{and} \quad F(t) = \frac{2e^{\frac{1}{2}t}}{e^t + 1},$$

respectively.

It has been known that the identity

$$(-1)^n(P_n(x) + 1)^m = (-1)^m(P_m(-x) + 1)^n \tag{4}$$

holds for the Bernoulli and Euler polynomials (see [2, 3]). Moreover, applying (4), recurrence relations for the corresponding Bernoulli and Euler numbers, defined respectively by $B_m = B_m(0)$ and $E_m = 2^m E_m(1/2)$, can be derived (see [6, 7, 8] for different approaches). We refer to [1, p. 803] for a good account of properties of Bernoulli and Euler polynomials and the associated Bernoulli and Euler numbers.

The object of this paper is to obtain a class of identities involving $P_m(x)$ as defined in (1) that generalizes (4). Our main result is formula (8) in Theorem 1, in which, curiously, the Beta function will play a role. New recurrence relations for the Bernoulli and Euler numbers are derived, including as a special case a formula of Gelfand [4] as cited in MR 43#7399.

2. A CLASS OF IDENTITIES

Before giving the main result in the following Theorem 1, we state two lemmas, of which the first has been given in [8], and can be verified by a simple application of the Leibniz rule.

Lemma 1: *For an integer $n \geq 0$, if $f(t,x)$ is nth differentiable in t for a fixed x, then*

$$f^{(n)}(t,x)e^t = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (f(t,x)e^t)^{(k)}, \tag{5}$$

where all derivatives are taken with respect to t.

We denote for an integer $r \geq 1$ by $\langle m \rangle_r = m(m+1)\cdots(m+r-1)$ the rising factorial, and as a convention $\langle m \rangle_0 = 1$. The Beta function $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ (see [1], p. 258), as well as the gamma function $\Gamma(m)$ in some exceptional case, plays a role.

Lemma 2: *If m, $n \geq 0$ are integers, then for an integer $r \geq 1$*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\langle m+k+1 \rangle_r} = \frac{B(m+1, n+r)}{(r-1)!}, \tag{6}$$

and for $m = n$

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1/2)^k}{\langle m+k+1 \rangle_r} = \frac{(1/2)^{m+1}}{(r-1)!} B(\frac{r}{2}, m+1). \tag{7}$$

Proof: We have (6) by evaluating the expression

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{x^{m+k+r}}{\langle m+k+1 \rangle_r} \equiv \frac{1}{(r-1)!} \int_0^x t^m (1-t)^n (x-t)^{r-1}\, dt$$

at $x = 1$. For $m = n$, the summation in (7) can be written as

$$\frac{m!}{(m+r)!} F(-m, m+1, m+r+1, 1/2),$$

where F denotes the Gauss hypergeometric function (see [1, p. 556]), and so (7) follows from formula 15.1.26 in [1].

Theorem 1: If m, $n \geq 0$ are integers, then for an integer $r \geq 1$

$$\begin{aligned} (-1)^n \sum_{k=0}^{m} \binom{m}{k} \frac{P_{n+k+r}(x)}{\langle n+k+1 \rangle_r} - (-1)^{m+r} \sum_{k=0}^{n} \binom{n}{k} \frac{P_{m+k+r}(-x)}{\langle m+k+1 \rangle_r} \\ = -\frac{1}{(r-1)!} \int_0^1 (1-s)^m s^n P_{r-1}(x+s)\, ds. \end{aligned} \tag{8}$$

Proof: Let $g(t,x)$ be the generating function given in (1). We define

$$f(t,x)=\frac{1}{t^r}\left[g(t,x)-\sum_{j=0}^{r-1}P_j(x)\frac{t^j}{j!}\right],$$

so that

$$f(t,x)=\sum_{j=r}^{\infty}P_j(x)\frac{t^{j-r}}{j!}=\sum_{j=0}^{\infty}\frac{P_{j+r}(x)}{\langle j+1\rangle_r}\frac{t^j}{j!}.$$

Then

$$f(t,x)e^t=\frac{1}{t^r}g(t,x+1)-\sum_{j=0}^{r-1}\frac{P_j(x)}{j!}\frac{e^t}{t^{r-j}}. \tag{9}$$

To expand the right-hand side of (9) in terms of powers of t, we write

$$\begin{aligned}\frac{1}{t^r}g(t,x+1)=&\left\{\frac{P_0(x+1)}{t^r}+\frac{1}{t^{r-1}}\frac{P_1(x+1)}{1!}+\cdots+\frac{1}{t}\frac{P_{r-1}(x+1)}{(r-1)!}\right\}\\ &+\sum_{m=0}^{\infty}\frac{P_{m+r}(x+1)}{\langle m+1\rangle_r}\frac{t^m}{m!},\end{aligned}$$

and write for each $0\le j\le r-1$ the corresponding term in the remaining summation as

$$\begin{aligned}\frac{P_j(x)}{j!}\frac{e^t}{t^{r-j}}=&\frac{P_j(x)}{j!}\left[\left\{\frac{1}{t^{r-j}}+\frac{1}{t^{r-j-1}}+\cdots+\frac{1}{(r-j-1)!\,t}\right\}\right.\\ &\left.+\sum_{m=0}^{\infty}\frac{1}{\langle m+1\rangle_{r-j}}\frac{t^m}{m!}\right].\end{aligned}$$

The coefficient of $1/t^m$ $(1\le m\le r)$ in the resulting expansion vanishes as it follows from (3) that

$$\frac{P_{r-m}(x+1)}{(r-m)!}-\sum_{j=0}^{r-m}\frac{P_j(x)}{j!\,(r-m-j)!}=0,$$

and so

$$f(t,x)e^t=\sum_{m=0}^{\infty}\left\{\frac{P_{m+r}(x+1)}{\langle m+1\rangle_r}-\sum_{j=0}^{r-1}\frac{P_j(x)}{j!\langle m+1\rangle_{r-j}}\right\}\frac{t^m}{m!}.$$

We now invoke Lemma 1 and substitute those obtained into (5). Then the left-hand side becomes

$$f^{(n)}(t,x)e^t = \left(\sum_{m=0}^{\infty} \frac{P_{m+n+r}(x)}{\langle m+n+1\rangle_r} \frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\right) = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}\frac{P_{n+k+r}(x)}{\langle n+k+1\rangle_r}\right)\frac{t^m}{m!},$$

and the right-hand side becomes

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{m=0}^{\infty}\left\{\frac{P_{m+k+r}(x+1)}{\langle m+k+1\rangle_r} - \sum_{j=0}^{r-1}\frac{P_j(x)}{j!\langle m+k+1\rangle_{r-j}}\right\}\frac{t^m}{m!}.$$

Comparing the coefficient of t^m $(m \geq 0)$ using (6) and the second formula in (2),

$$(-1)^n\sum_{k=0}^{m}\binom{m}{k}\frac{P_{n+k+r}(x)}{\langle n+k+1\rangle_r} - (-1)^{m+r}\sum_{k=0}^{n}\binom{n}{k}\frac{P_{m+k+r}(-x)}{\langle m+k+1\rangle_r}$$

$$= -\sum_{j=0}^{r-1}\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\langle m+k+1\rangle_{r-j}}\right\}\frac{P_j(x)}{j!}$$

$$= -\frac{m!}{(r-1)!}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{P_j(x)}{\langle n+r-j\rangle_{m+1}}.$$

Finally substituting $r-1$ for m in (3) and multiplying both sides by s^n, then integrating $m+1$ times over $(0,s)$ and evaluating the resulting equality at $s=1$, we have

$$\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{P_j(x)}{\langle n+r-j\rangle_{m+1}} = \frac{1}{m!}\int_0^1 (1-s)^m s^n P_{r-1}(x+s)\,ds, \tag{10}$$

from which the theorem follows.

As shown by (10), the right-hand side of (8) may as well be expressed by a summation. The present form puts m and n in perspective. Moreover, expanding $P_{r-1}(x+s)$ as a polynomial in s using (3), the integral in (10) can be written in a summation in terms of Beta functions.

The special case of Theorem 1 when $r=1$ is particularly interesting as the right-hand side of (8) is now a constant. It becomes Gelfand's formula cited in §1 when these $P_j(x)$ in (8) are the Bernoulli numbers B_j. The formula (4) is obtained by repeatedly differentiating both sides of (8) using the first formula in (2). Setting $m=n$ in (8) and using

$$(-1)^m B_m(-x) = mx^{m-1} + B_m(x), \quad (-1)^m E_m(-x) = 2x^m - E_m(x) \tag{11}$$

to replace the polynomials in $-x$ on the left-hand side of (8) by the same polynomials in x, Theorem 1 takes the following special form.

Corollary 1: For $m \geq 1$,

$$2\sum_{j=0}^{[(m-1)/2]} \binom{m}{2j+1} \frac{B_{2m-2j}(x)}{2m-2j} = x^m(x-1)^m - (-1)^m B(m+1,m+1); \tag{12}$$

$$2\sum_{j=0}^{[m/2]} \binom{m}{2j} \frac{E_{2m-2j+1}(x)}{2m-2j+1} = 2\int_0^x t^m(t-1)^m\,dt - (-1)^m B(m+1,m+1). \tag{13}$$

Here $[s]$ denotes the integer part of a real number s. We note that those Bernoulli polynomials in (12) have only even indices, and those Euler polynomials in (13) have only odd indices.

3. RECURRENCE RELATIONS

Recurrence relations for Bernoulli and Euler numbers, as well as Genocchi numbers defined as in [5] by $G_0 = 0$ and $G_n = nE_{n-1}(0)$ $(n = 1, 2, \cdots)$, can be obtained from (8) when x is assumed the value 0 or 1/2. We shall consider only the case $m = n$, which seems particularly interesting.

Theorem 2: *For integers $m \geq 0$ and $r \geq 1$, r odd*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+r}}{\langle m+k+1\rangle_r} = -\frac{1}{2(r-1)!}\int_0^1 (1-s)^m s^m B_{r-1}(s)\,ds; \tag{14}$$

$$\begin{aligned}(-1)^m \sum_{k=0}^{m} \binom{m}{k} &\frac{B_{m+k+r}}{2^{m+k}\langle m+k+1\rangle_r} \\ &= \frac{1}{(r-1)!}\left\{\frac{m!}{2^{2m+1}}\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}+m)} - \int_0^1 (1-s)^m s^m B_{r-1}(2s)\,ds\right\}.\end{aligned} \tag{15}$$

Proof: Let those $P_j(x)$ in (8) be Bernoulli polynomials. Setting $x = 0$, (14) follows immediately. Setting $x = 1/2$, after using the first formula in (11) to replace those polynomials in $-1/2$ by the same polynomials in $1/2$, we obtain (15) as $B_j(1/2) = (2^{1-j}-1)B_j$. In applying (7), the gamma functions are used to adapt the case $r = 1$. The last term on the right hand side is obtained by the multiplication formula $B_{r-1}(2s) = 2^{r-2}\{B_{r-1}(s) + B_{r-1}(s+\frac{1}{2})\}$.

Theorem 3: *For integers $m \geq 0$ and $r \geq 2$, r even*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{G_{m+k+r}}{\langle m+k+1\rangle_r} = -\frac{1}{2(r-2)!}\int_0^1 (1-s)^m s^m E_{r-2}(s)\,ds. \tag{16}$$

Proof: Let those $P_j(x)$ in (8) be Euler polynomials. Replacing r by $r-1$ and setting $x = 0$, (16) follows immediately.

Theorem 4: *For integers $m \geq 0$ and $r \geq 2$, r even*

$$\begin{aligned}(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+r}}{2^{m+k} \langle m+k+1 \rangle_r} \\ = \frac{1}{(r-1)!} \left\{ \frac{1}{2^{2m+1}} B\left(\frac{r}{2}, m+1\right) - 2^{r-1} \int_0^1 (1-s)^m s^m E_{r-1}\left(\frac{1}{2}+s\right) ds \right\}.\end{aligned} \tag{17}$$

Proof: Let these $P_j(x)$ in (8) be Euler polynomials. Setting $x = 1/2$, we use the second formula in (11) to replace those polynomials in $-1/2$ by the same polynomials in $1/2$, and derive (17) similarly to that of (15).

In proving Theorems 2, 3 and 4 it becomes obvious that if r is in a parity different from the one stated in the theorems, the right hand sides of these formulas derived therein vanish. For (14) and (16), this may be deduced directly from the representing integrals. As for (15) and (17), we obtain by (10) the following equalities.

Corollary 2: *For integers $m \geq 0$ and $r \geq 2$, r even*

$$\sum_{j=0}^{r-1} \binom{r-1}{j} \frac{2^{r-j} B_j}{\langle r+m-j \rangle_{m+1}} = \frac{1}{2^{2m}} \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}+m)}.$$

Corollary 3: *For integers $m \geq 0$ and $r \geq 1$, r odd*

$$\sum_{j=0}^{r-1} \binom{r-1}{j} \frac{2^{r-j} E_j}{\langle r+m-j \rangle_{m+1}} = \frac{1}{2^{2m}\, m!} B\left(\frac{r}{2}, m+1\right).$$

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The idea goes as follows: Multiply both sides of (8) by $\frac{y^m}{m!} \frac{z^n}{n!} w^r$ and take the triple sums $\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}$ on the left hand-side and $\sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}$, in which r starts with 1, on the right-hand side. These two coincide and both equal to $\frac{w}{w-y+z} e^{xw}(e^{y-w/2} - e^{z+w/2})F(w)$. This also gives a direct proof of (4) for general $P_m(x)$ as for $r = 0$ the left hand-side of the equality vanishes.

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