THE ASYMPTOTIC GROWTH RATE OF RANDOM FIBONACCI TYPE SEQUENCES II

Hei-Chi Chan
Mathematical Science Program, University of Illinois at Springfield, Springfield, IL 62703-5407
e-mail: chan.hei-chi@uis.edu

(Submitted September 2003-Final Revision March 2004)

ABSTRACT

In this paper, we use ergodic theory to compute the aysmptotic growth rate of a family of random Fibonacci type sequences. This extends the result in [2]. We also prove some Lochs-type results regarding the effectiveness of various number theoretic expansions.

1. INTRODUCTION

Let $x \in [0, 1)$ and k be a fixed integer greater than or equal to 2. As we shall show in the next section, we can write x as

$$x = \cfrac{k^{-a_1}}{1 + \cfrac{(k-1)k^{-a_2}}{1 + \cfrac{(k-1)k^{-a_3}}{1 + \cdots}}} \equiv [a_1, a_2, a_3, \cdots]_k, \tag{1}$$

where the "digits" $a_m = a_m(x)$ are natural numbers. The main result of this paper is the following theorem:

Theorem 1: *For each $x \in [0, 1)$, we associate with it an infinite sequence of natural numbers $\{a_1, a_2, \cdots\}$ through (1). Consider the random Fibonacci type sequences, $\{Q_m\}$, defined by $Q_{-1} = 0$, $Q_0 = 1$, $a_0 = 0$, and*

$$Q_m = k^{a_m} Q_{m-1} + (k-1)k^{a_{m-1}} Q_{m-2}, \tag{2}$$

for $m \geq 1$. Then, for almost all x with reference to the Lebesgue measure, the asymptotic growth rate of Q_n is given by

$$\beta^*(k) \equiv \lim_{n\to\infty} \frac{1}{n} \ln Q_n = c_k \int_0^1 \frac{\ln(1/x)}{(1+(k-1)x)(k+(k-1)x)}\, dx \leq c_k \frac{3k-1}{2k(2k-1)}, \tag{3}$$

where $c_k = (k-1)^2/\log(k^2/(2k-1))$.

The case of $k = 2$ was first proved in [2] and it was motivated by the work of Viswanath [19]; see also [7, 21]. For an interesting introduction, see Peterson's article [14]. Theorem 1 is a generalization of the case of $k = 2$, and its proof (cf. Section 4) uses the same strategy used in [2]. Precisely, it makes use of an interval map, T_k (defined in Section 2) and its *ergodicity* (cf. Section 3). A key ingredient of the proof of Theorem 1 is the *invariant probability density* of the map T_k (cf. equation (15)). The explicit form of the invariant probability density was kindly pointed out by the referee of [2]. It should be noted that, given an interval map, in general, it is a difficult task to obtain the explicit form of its invariant probability density; for some non-trivial examples, see, e.g., [5, 13, 17] (see also [6, 10, 18] and the references quoted therein). As an application of the ergodicity of T_k, we prove a Khintchin-type result in Section 3.

In Section 5, we compute the *entropy* of T_k. In brief, the entropy of an interval map reflects the amount of randomness generated by the map. It is also an isomorphism invariant, so that isomorphic transformations would have the same entropy. For an introduction, see, e.g., [1]. As an application, we prove results regarding the effectiveness of various number theoretic expansions. For example, consider a real number $x \in [0, 1)$. On the one hand, we can write x in the decimal expansion, i.e., $x = 0.d_1 d_2 d_3 \cdots$. On the other hand, we write $x = [a_1, a_2, a_3, \cdots]_2$. For notation, see (1). Suppose we are given the first n digits of the decimal expansion of x. Then these n digits determine $m_{D2}(n, x)$ digits of the expansion $[a_1, a_2, a_3, \cdots]_2$. We shall prove that, for almost all $x \in [0, 1)$ with respect to the Lebesgue measure, we have

$$\lim_{n\to\infty} \frac{m_{D2}(n, x)}{n} = 1.41826 \cdots . \tag{4}$$

In the rest of the paper, k always denotes a fixed integer greater than or equal to 2.

2. THE INTERVAL MAP T_k

The goal of this section is to define the interval map T_k and set up the preliminaries that are needed for the rest of the paper.

First, we prove that

Lemma 1: *For all $x \in [0, 1)$, we have $x = [a_1, a_2, a_3, \cdots]_k$, where a_m are natural numbers.*

Proof: For $x \in [0, 1)$, we can find a natural number a_1 such that $k^{-a_1-1} < x \le k^{-a_1}$. We can also write this as

$$\frac{k^{-a_1}}{1 + (k-1)} < x \le \frac{k^{-a_1}}{1 + 0}.$$

This means we can find a unique $x_1 \in [0, 1)$ such that

$$x = \frac{k^{-a_1}}{1 + (k-1)\, x_1}.$$

Since $x_1 \in [0, 1)$, we can repeat the same iteration and obtain

$$x = \cfrac{k^{-a_1}}{1 + \cfrac{(k-1)k^{-a_2}}{1 + \cfrac{(k-1)k^{-a_3}}{1 + \cdots}}} = [a_1, a_2, a_3, \cdots]_k.$$

□

As an example, we have $\pi - 3 = [2, 0, 1, 0, 0, 0, 1, 1, \cdots]_2 = [1, 1, 0, 0, 1, 0, 1, 1, \cdots]_5$.

Next, we define T_k:

Definition 1: *Define the interval map* $T_k : [0,1) \to [0,1)$ *as follows: for* $x = 0$, $T_k 0 \equiv 0$*; for* $x \neq 0$, $T_k x = T_k[a_1, a_2, a_3, \cdots]_k \equiv [a_2, a_3, a_4, \cdots]_k$. *There is another way to define* T_k *for* $x \neq 0$. *With* $\lfloor \bullet \rfloor$ *denoting the floor function, define*

$$a(x) \equiv \left\lfloor \frac{\log(1/x)}{\log k} \right\rfloor .$$

Then we have

$$T_k x = \frac{1}{k-1}\left(\frac{k^{-a(x)}}{x} - 1\right). \tag{5}$$

To see (5), observe that $x = k^{-a(x)}/(1 + (k-1)\, T_k x)$. Below are the graphs for T_2 and T_5:

With the above understood, we define the following recursions:

Definition 2: *For all* $x = [a_1, a_2, \cdots]_k \in [0,1)$, *define*

$$\begin{aligned} P_m(x) &= k^{a_m(x)} P_{m-1}(x) + (k-1) k^{a_{m-1}(x)} P_{m-2}(x), \quad k \geq 2 \\ Q_m(x) &= k^{a_m(x)} Q_{m-1}(x) + (k-1) k^{a_{m-1}(x)} Q_{m-2}(x), \quad k \geq 1 \end{aligned} \tag{6}$$

where $P_0 = 0,\ P_1 = 1,\ Q_{-1} = 0,\ Q_0 = 1$ *and* $a_0(x) \equiv 0$.

Note that the second equation in (6) is the recursion in (2). Note that P_m and Q_m depend on k, even though the explicit dependence is suppressed in the present notation. Note also that (6) is related to continued fractions $[a_1, a_2, \cdots]_k$. Define the compact notation

$$\cfrac{b_1}{d_1 + \cfrac{b_2}{d_2 + \cdots}} \equiv \frac{b_1|}{|d_1} + \frac{b_2|}{|d_2} + \cdots .$$

With this understood, we can write

$$[a_1, a_2, a_3, \cdots]_k = \frac{k^{-a_1}|}{|\ 1} + \frac{(k-1)\, k^{-a_2}|}{|\quad 1} + \frac{(k-1)\, k^{-a_3}|}{|\quad 1} + \cdots .$$

Standard induction shows that

$$\frac{k^{-a_1}|}{|\ \ 1} + \frac{(k-1)\,k^{-a_2}|}{|\qquad 1} + \cdots + \frac{(k-1)\,k^{-a_n}|}{|1+(k-1)\,t} = \frac{P_n + t\,(k-1)\,k^{a_n} P_{n-1}}{Q_n + t\,(k-1)\,k^{a_n} Q_{n-1}}, \tag{7}$$

for $0 \le t \le 1$, and

$$P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x) = (-1)^n\,(k-1)^{n-1} k^{a_1} \cdots k^{a_{n-1}}. \tag{8}$$

By combining Lemma 1 and (7), we have, for $x \in [0,1)$,

$$x = \frac{P_n(x) + t\,(k-1)\,k^{a_n(x)} P_{n-1}(x)}{Q_n(x) + t\,(k-1)\,k^{a_n(x)} Q_{n-1}(x)}, \tag{9}$$

where $t = T_k^n x$ (i.e., iterating the map T_k for n times). Taking $t = 0$ in (9) gives the nth approximation of x:

$$[a_1, a_2, \cdots, a_n]_k \equiv \frac{P_n(x)}{Q_n(x)}. \tag{10}$$

Furthermore, we can show by induction that $P_n(x) = Q_{n-1}(T_k x)$. In fact, a similar result also holds for the *deformation* of P_n and Q_n. Precisely, we have:

Definition 3: *For $n = 1, 2, \cdots$, let $t = T_k^n x$ and define*

$$\begin{aligned} A_n(x) &= P_n(x) + t\,(k-1)\,k^{a_n(x)} P_{n-1}(x), \\ B_n(x) &= Q_n(x) + t\,(k-1)\,k^{a_n(x)} Q_{n-1}(x). \end{aligned} \tag{11}$$

Note that (9) and (11) imply $x = A_n(x)/B_n(x)$. By induction, we can show that

$$A_n(x) = B_{n-1}(T_k x). \tag{12}$$

By combining (8)-(10), we have

$$|x - [a_1, \cdots, a_n]_k| = \frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n\,(t^{-1}Q_n + (k-1)k^{a_n} Q_{n-1})}, \tag{13}$$

where $t = T_k^n x$. Note that this equation, which measures the difference between x and its nth approximation, is the key ingredient of the following estimate:

Lemma 2: *For all $x \in [0,1)$, we have* $|x - [a_1, \cdots, a_n]_k| \le ((k-1)/k)^n$.

Remarks: This implies the limit $x = \lim_{n\to\infty} [a_1, a_2, \cdots, a_n]_k$ exists, as $(k-1)/k < 1$.

Proof: Denote $\lambda = (k-1)/k$. By using (13) and the fact that $t^{-1} \ge 1$, we have

$$|x - [a_1, \cdots, a_n]_k| \le \frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n\,(Q_n + (k-1)\,k^{a_n} Q_{n-1})} \equiv s_n.$$

We claim that $s_n \leq \lambda\, s_{n-1}$. Indeed, by (6), $Q_n + (k-1)\, k^{a_n} Q_{n-1} \geq k\, k^{a_n} Q_{n-1}$, therefore

$$s_n \leq \lambda \left(\frac{(k-1)^{n-1} k^{a_1+\cdots+a_{n-1}}}{Q_n Q_{n-1}} \right) \leq \lambda \left(\frac{(k-1)^{n-1} k^{a_1+\cdots+a_{n-1}}}{Q_{n-1}\, (Q_{n-1} + (k-1)\, k^{a_{n-1}} Q_{n-2})} \right) = \lambda\, s_{n-1}.$$

In obtaining the second inequality, we have used the fact that $Q_n \geq Q_{n-1} + (k-1)\, k^{a_{n-1}} Q_{n-2}$. This proves the claim. By direct computation, we have $s_1 \leq \lambda\, k^{-a_1} \leq \lambda$. This, with $s_n \leq \lambda\, s_{n-1}$, shows that $s_n \leq \lambda^n$. This proves the lemma. □

Finally, we define the *fundamental cylinders*. They are a particular partition that is natural to continued fractions of type (1). First, let $t \in [0,1)$ and define $\psi_{a_1 \cdots a_n}(t)$ by

$$\psi_{a_1 \cdots a_n}(t) = \frac{k^{-a_1}|}{|\ \ 1} + \frac{(k-1)\, k^{-a_2}|}{|\qquad 1} + \cdots + \frac{(k-1)\, k^{-a_n}|}{|1 + (k-1)\, t}. \tag{14}$$

We define $\triangle_{a_1 \cdots a_n} = \{\psi_{a_1 \cdots a_n}(t);\ t \in [0,1)\}$ to be the *fundamental cylinder* of rank n. We also define $\triangle_a$ (i.e., the cylinders of rank one) to be the *atoms* of the partition.

A key property of the cylinders is that

Lemma 3: *Let l denote the Lebesgue measure; then* $l(\triangle_{a_1 \cdots a_n}) \leq ((k-1)/k)^n$.

Remark: this lemma implies that the class of all cylinders generates the σ-algebra $\mathcal{F}$ of Borel sets. Now, we turn to the proof of this lemma.

Proof: By direct computation, we have

$$l(\triangle_{a_1 \cdots a_n}) = |\psi_{a_1 \cdots a_n}(1) - \psi_{a_1 \cdots a_n}(0)| = \frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n\, (Q_n + (k-1)\, k^{a_n} Q_{n-1})} = s_n.$$

Note that, in the second equality, we have used (7) and mimicked the same tricks used in showing (13). The present lemma is proven by noting that $s_n \leq ((k-1)/k)^n$, as shown in the proof of the previous lemma. □

This concludes the preliminary set up and we are ready to study the ergodicity of T_k, to which we now turn.

3. THE ERGODICITY OF T_k

In this section, we prove that T_k is ergodic with respect to the following measure:

$$\mu_k(A) = c_k \int_A \frac{dx}{(1 + (k-1)x)(k + (k-1)x)}, \tag{15}$$

where A is an element of the σ-algebra $\mathcal{F}$ of Borel sets. Here, the normalization c_k, defined in Theorem 1, is chosen so that $\mu_k([0,1]) = 1$. μ_2 was first discovered by the author in [2]. Thanks to the referee of the same paper who pointed out the explicit expression of (15) for $k \geq 3$.

Since, for $x \in [0,1)$, the integrand in (15) is bounded between $c_k/(k\,(2k-1))$ and c_k/k, therefore, we can find constants d_L and d_U with $0 < d_L < d_U < \infty$, such that

$$d_L\ l(A) \leq \mu_k(A) \leq d_U\ l(A). \tag{16}$$

An important property of μ_k is that it is *preserved* by the interval map T_k; i.e., $\mu_k(T_k^{-1}A) = \mu_k(A)$ for every $A \in \mathcal{F}$. To establish this, it is enough to show that

Lemma 4: *For $t > 0$, we have* $\mu_k\left(T_k^{-1}[0,t]\right) = \mu_k\left([0,t]\right)$.

Proof: Let $\gamma = 1/k$ and define $W(m) = \log\left(1 + (k-1)\,\gamma^m\right) - \log\left(1 + \frac{(k-1)\,\gamma^m}{1+(k-1)\,t}\right)$. Then we have

$$\mu_k\left(T_k^{-1}[0,t]\right) = \mu_k\left(\bigcup_{m=0}^{\infty}\left[\frac{\gamma^m}{1+(k-1)\,t},\ \gamma^m\right]\right) = \sum_{m=0}^{\infty}\mu_k\left(\left[\frac{\gamma^m}{1+(k-1)\,t},\ \gamma^m\right]\right)$$

$$= \sum_{m=0}^{\infty}\int_{\gamma^m/(1+(k-1)\,t)}^{\gamma^m} d\mu_k$$

$$= \frac{c_k}{(k-1)^2}\sum_{m=0}^{\infty} W(m) - W(m+1)$$

$$= \frac{c_k}{(k-1)^2}\,W(0) = \frac{c_k}{(k-1)^2}\log\left(\frac{k(1+(k-1)\,t)}{k+(k-1)\,t}\right).$$

Direct computation shows that the last expression is $\mu_k([0,t])$. $\square$

With this understood, we prove the following:

Theorem 2: *T_k is ergodic with respect to μ_k.*

Proof: Our proof follows the same strategy used in [2]. It is the strategy used by Billingsley [1] to prove the ergodicity of the continued fraction map; see also [3, 6, 9, 10, 15]. All we need to do is to show that if $A \in \mathcal{F}$ is such that $T^{-1}A = A$ and $\mu_k(A) > 0$, then $\mu_k(A) = 1$. To proceed, for fixed $a_1, \cdots, a_n$, let us denote $\psi_{a_1\cdots a_n}$ and $\triangle_{a_1\cdots a_n}$, which are defined in (14) and the sentence right after it, by $\psi_{(n)}$ and $\triangle_{(n)}$ respectively.

First, we want to prove that, for $A \in \mathcal{F}$, we have

$$l\left(T_k^{-n}A \cap \triangle_{(n)}\right) \geq \frac{1}{k}\,l(A)\,l(\triangle_{(n)}). \tag{17}$$

To this end, we first consider A being an interval $[x,y)$, where $0 \leq x < y \leq 1$. Observe that

$$\begin{aligned}\frac{l\left(T_k^{-n}A \cap \triangle_{(n)}\right)}{l\left(\triangle_{(n)}\right)} &= \frac{\left|\psi_{(n)}(y) - \psi_{(n)}(x)\right|}{\left|\psi_{(n)}(1) - \psi_{(n)}(0)\right|} \\ &= l(A)\underbrace{\left(\frac{Q_n(Q_n+(k-1)\,k^{a_n}Q_{n-1})}{(Q_n+x\,(k-1)\,k^{a_n}Q_{n-1})(Q_n+y\,(k-1)\,k^{a_n}Q_{n-1})}\right)}_{\equiv\ h(x,y)}\end{aligned} \tag{18}$$

Note that, in obtaining the second equality, we have used the fact that $l(A) = |y - x|$ and have mimicked similar tricks that are used in the proof of Lemma 2 and 3.

Since $h(x,y) \geq h(1,1) = 1/k$, therefore (18) implies that (17) holds for A being an interval. ¿From this, with the inner and outer regularity of measures, it follows that (17) holds also if A is a disjoint union of intervals, and hence holds for any $A \in \mathcal{F}$.

Because of (16), inequality (17) implies an analogous result for $\mu_k(A)$; precisely, for $A \in \mathcal{F}$, we have

$$\mu_k\left(T_k^{-n} A \cap \triangle_{(n)}\right) \geq C^* \mu_k(A)\, \mu_k(\triangle_{(n)}) \tag{19}$$

for some positive constant C^*. Now, this inequality implies the theorem. Indeed, let A be such that $T_k^{-1} A = A$ and suppose $\mu_k(A) > 0$. Then, inequality (19) implies that $\mu_k(A \cap E) \geq C^* \mu_k(A) \mu_k(E)$ holds for finite disjoint unions E of fundamental cylinders; since these sets generate $\mathcal{F}$, we have $\mu_k(A \cap E) \geq C^* \mu_k(A) \mu_k(E)$ for any $E \in \mathcal{F}$. Taking $E = A^c$, the complement of A, we see that $\mu_k(A^c) = 0$ and so $\mu_k(A) = 1$. □

With the ergodicity of T_k proven, we can apply the Ergodic Theorem to prove:

Theorem 3: *For almost all $x = [a_1, a_2, \cdots]_k \in (0,1)$, we have*

$$\begin{aligned} \alpha^*(k) &\equiv \lim_{n\to\infty} \log\left(k^{a_1+\cdots+a_n}\right)^{1/n} \\ &= \frac{c_k \log k}{(k-1)^2} \sum_{m=0}^{\infty} \log\left(1 + \frac{(k-1)^3}{k^{m+2} + 2(k-1)k + \dfrac{(k-1)^2}{k^m}}\right)^m . \end{aligned} \tag{20}$$

Proof: To prove this Khintchin-type result, we proceed as follows. See Finch's book [8] on the orginal Khintchin constant. Again, let $\gamma = 1/k$. Consider the integer-valued function $a(x)$ which gives the first digit of x; cf. Definition 1. Note that $a(x) = m$ whenever $k^{-(1+m)} < x \leq k^{-m}$. Consider, then, the following integral

$$\begin{aligned} \int_0^1 a(x)\, d\mu_k &= \sum_{m=0}^{\infty} m \int_{\gamma^{m+1}}^{\gamma^m} d\mu_k \\ &= \frac{c_k}{(k-1)^2} \sum_{m=0}^{\infty} m \log \underbrace{\left(1 + \frac{(k-1)^3}{k^{m+2} + 2(k-1)k + \dfrac{(k-1)^2}{k^m}}\right)}_{\equiv H_m} . \end{aligned} \tag{21}$$

Since

$$\log H_m \leq \log\left(1 + \frac{k^3}{k^{m+2}}\right) \leq \frac{k}{k^m},$$

therefore $\sum m \log H_m \leq (1 - 1/k)^{-2}$; i.e., the series is convergent. Therefore, we can apply the Ergodic Theorem: for almost all x, we have

$$\alpha^*(k) = \lim_{n\to\infty} \log\left(k^{a_1+\cdots+a_n}\right)^{1/n} = \log k \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} a(T_k^m x) = \log k \int_0^1 a(x)\, d\mu_k. \tag{22}$$

This, with (21), completes the proof of the theorem. □

As examples, we have $\alpha^*(2) = 0.97693\cdots$, $\alpha^*(3) = 0.95476\cdots$ and $\alpha^*(100) = 0.80902\cdots$.

4. PROOF OF THEOREM 1

We follow the strategy used in [2]; see also [1, 3, 6, 18]. Recall that $x = A_n(x)/B_n(x)$; cf. Definition 3 and the sentence immediately following it. Similarly, we have $T_k^m x = A_{n-m}(T_k^m x)/B_{n-m}(T_k^m x)$. With this understood, we have

$$\prod_{i=0}^{n-1} T_k^i x = \frac{A_n(x)}{B_n(x)} \frac{A_{n-1}(T_k x)}{B_{n-1}(T_k x)} \cdots \frac{A_1(T_k^{n-1} x)}{B_1(T_k^{n-1} x)} = \frac{1}{B_n(x)}. \tag{23}$$

Because of (12) we have almost all factors canceled, except $B_n(x)$ and $A_1(T_k^{n-1}x) = 1$.

By using the definition of $B_n(x)$ (cf. 11) and the fact that $0 \le T_k^n x < 1$, we have $Q_n \le B_n \le kQ_n$. This, with (23), implies

$$\frac{1}{kQ_n(x)} \le \prod_{i=0}^{n-1} T_k^i x \le \frac{1}{Q_n(x)}.$$

This shows that $-\log k - \log Q_n(x) \le \sum \log\, T_k^i x \le -\log Q_n(x)$. By the Ergodic Theorem, we have, for almost all x,

$$\beta^*(k) = \lim_{n\to\infty} \frac{1}{n} \log Q_n = -\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\, T_k^i x = \int_0^1 \log(1/x)\, d\mu_k. \tag{24}$$

To find the upper bound stated in Theorem 1, we observe that, for $x \in [0, 1]$,

$$0 \le \frac{1}{(1 + (k-1)\,x)(k + (k-1)\,x)} \le -\frac{2(k-1)}{k\,(2k-1)} x + \frac{1}{k}.$$

This, with $\int_0^1 (ax + b) \log(1/x)\, dx = b + (a/4)$, where a and b are constants, implies

$$\int_0^1 \log(1/x)\, d\mu_k \le c_k \int_0^1 \log(1/x) \left(-\frac{2(k-1)}{k\,(2k-1)}\, x + \frac{1}{k} \right) dx = c_k \frac{3k-1}{2k(2k-1)}.$$

This proves Theorem 1. □

As examples, we have $\beta^*(2) = 1.30022\cdots$ (proven in [2]), $\beta^*(3) = 1.45799\cdots$ and $\beta^*(100) = 2.90446\cdots$.

5. THE ENTROPY OF T_k

The main result of this section is

Theorem 4: *The entropy of T_k, denoted by $h(T_k)$, on the unit interval with respect to the measure μ_k is given by $h(T_k) = 2\,\beta^*(k) - \alpha^*(k) - \log(k-1)$, where $\beta^*(k)$ and $\alpha^*(k)$ are defined in Theorems 1 and 3 respectively.*

Proof: Let $\triangle_{(n)}(x)$ be the fundamental cylinder of rank n that contains the point x. Then, by using the same tricks for proving Lemma 3, we have

$$l(\triangle_{(n)}(x)) \frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n\left(Q_n+(k-1)\, k^{a_n} Q_{n-1}\right)}.$$

Since $Q_n \leq Q_n+(k-1)\, k^{a_n} Q_{n-1} \leq Q_n+(k-1)\, Q_n = kQ_n$, we have

$$\frac{1}{k}\left(\frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n^2}\right) \leq l(\triangle_{(n)}(x)) \leq \left(\frac{(k-1)^n k^{a_1+\cdots+a_n}}{Q_n^2}\right).$$

This implies, for almost all x,

$$\begin{aligned}\lim_{n\to\infty} \frac{-\log l(\triangle_{(n)}(x))}{n} &= 2 \lim_{n\to\infty} \frac{1}{n} \log Q_n - \lim_{n\to\infty} \log \left(k^{a_1+\cdots+a_n}\right)^{1/n} - \log(k-1) \\ &= 2\,\beta^*(k) - \alpha^*(k) - \log(k-1),\end{aligned}$$

and we are done. □

Remarks: We could have used the *Rohlin Entropy Formula*; cf. [16]; see also [15]. All we have to do is to check *Rényi's condition*: we can show that

$$\frac{\left|\psi'_{a_1\cdots a_n}(t)\right|}{\left|\psi'_{a_1\cdots a_n}(r)\right|} = \left(\frac{Q_n + r\,(k-1)\, k^{a_n} Q_{n-1}}{Q_n + t\,(k-1)\, k^{a_n} Q_{n-1}}\right)^2 \leq \left(\frac{Q_n + (k-1)\, k^{a_n} Q_{n-1}}{Q_n}\right)^2 \leq k^2.$$

Here, $\psi'_{a_1\cdots}(t)$ denotes the derivative of $\psi_{a_1\cdots}(t)$ with respect to t. The first equality is due to (7) and (14). The first inequality is obtained when setting $r=1$ and $t=0$. In the last inequality, again, we have used $Q_n+(k-1)\, k^{a_n} Q_{n-1} \leq kQ_n$. So the ratio of the derivatives is bounded and Rényi's condition is satisfied.

Now, by using Rohlin's formula, we have $(T_k(x) \equiv T_k x)$

$$\begin{aligned}h(T_k) &= \int_0^1 \log |T_k'(x)|\; d\mu_k \\ &= \int_0^1 \log\left(\frac{k^{-a(x)}}{(k-1)x^2}\right) d\mu_k \\ &= \int_0^1 (2\,\log(1/x) - a(x)\log k - \log(k-1))\; d\mu_k. \\ &= 2\,\beta^*(k) - \alpha^*(k) - \log(k-1).\end{aligned}$$

Note that the derivative of $T_k(x)$ is calculated using (5). The last line is due to (22) and (24).

As examples, we have, $h(T_2) = 1.62352\cdots$, $h(T_3) = 1.26808\cdots$ and $h(T_{100}) = 0.40478\cdots$. This implies the maps T_2, T_3 and T_{100} are not isomorphic to each other (because entropy is an isomorphism invariant; e.g., see Theorem 6.1.7 in [6]). This observation leads to

Conjecture 1: *T_k is not isomorphic to T_l if $k \neq l$.*

Lastly, we prove some results regarding the effectiveness of various number theoretic expansions. In the 60s, Lochs proved the following striking result. Write $x \in [0,1)$ as a regular continued fraction (RCF); i.e.,

$$x = \frac{1\ |}{|\ c_1} + \frac{1\ |}{|\ c_2} + \frac{1\ |}{|\ c_3} + \cdots .$$

Here, the digits c_m are integers greater than or equal to 1. Represent the same x by its decimal representation; i.e., $x = 0.d_1 d_2 d_3 \cdots$. Denote by $m_{DR}(n,x)$ the number of digits of the RCF that are determined by the first n digits of the decimal expansion of x. Here, the subscript DR stands for the two different number theoretic expansions ("D" stands for "decimal" and "R" for "regular " in RCF). Lochs proved the following [11]:

Theorem 5: *For almost all $x \in [0,1)$ with respect to the Lebesgue measure, we have*

$$\lim_{n\to\infty} \frac{m_{DR}(n,x)}{n} = \frac{6\ \log 2\ \log 10}{\pi^2} = 0.970270\cdots .$$

Remarkably, Bosma, Dajani and Kraaikamp (*Entropy and Counting Correct Digits*, Rapporten Mathematisch Instituut preprint, June 1999) generalized Lochs' result, comparing the rate of approximation of a wide range of number theoretic expansions (such as the alternating Lüroth expansion and the g-adic expansions). Quite recently, this result was beautifully extended by Dajani and Fieldsteel [4] to the most general setting, and based on this result, we shall prove results such as that of (4).

In order to prove (4), we need to know the interval maps that generate the expansions. The expansions of the type of (1), i.e., $[a_1, a_2, \cdots]_k$, are generated by iteration of T_k. The interval map that generates the decimal representation is well-known (e.g., cf. [6]): it is defined by $Sx = 10x \pmod 1$, where $x \in [0,1)$. Analogous to $\triangle_{a_1\cdots a_k}$, we define, $B_{d_1\cdots d_n}$, the *decimal cylinder* of order n. Here, $d_k \in \{0,1,\cdots,9\}$. Let $y = 0.d_1 d_2 \cdots d_n$ and $z = y + 10^{-n}$. Then, we define $B_{d_1\cdots d_n} = [y,z]$. We also define the *atoms* of this partition to be B_d.

It is well-known that S preserves the Lebesgue measure and its entropy is given by $\log 10$. Note that, cf. [4], $m_{D2}(n,x)$ in (4) can also be defined as

$$m_{D2}(n,x) = \sup\ \{m;\ B_{d_1\cdots d_n}(x) \subset \triangle_{a_1\cdots a_m}(x)\}.$$

Here, the subscript "2" stands for the expansion $[a_1,\cdots]_2$ (i.e., the case of $k=2$).

Next we introduce the notion of the *number theoretic fibered maps* (NTFMs), to which the result of Dajani and Fieldsteel applies. Precisely, a surjective map $U : [0,1) \to [0,1)$ is a NTFM if it satisfies ([4]; see also [18] and the preprint by Bosma, Dajani and Kraaikamp cited above):

1. there exists a finite or countable partition of intervals $P = \{P_\alpha;\ \alpha \in \mathcal{D}\}$ (here $\mathcal{D}$ is the digit set) such that U restricted to each atom of P is monotone, continuous and injective,
2. U is ergodic with respect to the Lebesgue measure l, and there exists a U invariant probability measure μ equivalent to l with bounded density.

Maps such as S and the continued fraction map ($T_{CF}(x) = 1/x - \lfloor 1/x \rfloor$) are NTFMs (cf. [4]). It is not hard to see that T_k, which is analogous to $T_{CF}(x)$, is also a NTFM. With this understood, consider NTFMs U and V on $[0,1)$, with invariant measures μ_U and μ_V (equivalent to the Lebesgue measure) and with partitions P and Q respectively. Denote by $P_{(n)}(x)$ the cylinder of rank n that contains $x \in [0,1)$ (a similar definition for $Q_{(m)}(x)$). Let $m(n,x) = \sup\{m;\ P_{(n)}(x) \subset Q_{(m)}(x)\}$. Suppose that $h(U) > 0$ and $h(V) > 0$. Then we have (Theorem 4 in [4]):

Theorem 6: *(Dajani and Fieldsteel, 2001) Under the conditions just stated, with respect to the Lebesgue measure, we have, for almost all $x \in [0,1)$,*

$$\lim_{n\to\infty} \frac{m(n,x)}{n} = \frac{h(U)}{h(V)}.$$

Apply this theorem with $U = S$ and $V = T_{CF}$ (the continued fraction map), we get back Lochs' result (Theorem 5), as $h(S) = \log 10$ and $h(T_{CF}) = \pi^2/(6\log 2)$.

Equation (4) also follows immediately from this beautiful theorem. Set $U = S$ and $V = T_2$. Since $h(S) = \log 10$ and $h(T_2) = 2\beta^*(2) - \alpha^*(2) = 1.62352\cdots$ (cf. the examples listed after the proof of Theorem 4), Theorem 6 implies, for almost all x,

$$\lim_{n\to\infty} \frac{m_{D2}(n,x)}{n} = \frac{h(S)}{h(T_2)} = 1.41826\cdots.$$

We can also compare the effectiveness of the expansion in (1) with different k. For example, write $x \in [0,1)$ as

$$x = [d_1, d_2, \cdots]_2 = [e_1, e_2, \cdots]_3.$$

Let $m_{23}(n,x)$ the number of digits of the expansion in $k = 3$ that are determined by the first n digits of the expansion in $k = 2$. Here, the subscript "23" stands the expansions for $k = 2$ and for $k = 3$. Then, by the same argument, we have

$$\lim_{n\to\infty} \frac{m_{23}(n,x)}{n} = \frac{h(T_2)}{h(T_3)} = 1.28029\cdots.$$

Remarks: By imitating Mayer [12] and Wirsing [20], one can derive not just the constant ($\beta^*(k)$) but also the *rate* of convergence to the constant. This will be addressed elsewhere.

ACKNOWLEDGMENTS

I would like to thank my colleagues Chung-Hsien Sung and Doug Woken for their valuable comments and support on this project. Part of this work was presented at the Conference on Ergodic Theory with Connections to Number Theory at the Oregon State University (July, 2003). I would like to thank the conference organizers (Pierre Arnoux, Bob Burton and Tom Schmidt) and the participants, especially, El Houcein El Abdalaoui, Vitali Bergelson, Reza Chamanara and Karma Dajani for their encouragement and interest in this work. I would also like to thank Cor Kraaikamp for email correspondence and encouragement. I would like to thank the referee of [2], who kindly pointed out the explicit form of the invariant measure in (15) for $k \geq 3$. Last but not least, I would like to thank the referee of the present paper for encouragement and suggestions.

REFERENCES

[1] P. Billingsley. *Ergodic Theory and Information.* John Wiley & Sons, New York, 1965.

[2] H. Chan. "The Asymptotic Growth Rate of Random Fibonacci Type Sequences." *The Fibonacci Quarterly* **43** (2005): 243-255.

[3] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai. *Ergodic Theory.* Springer-Verlag, New York, 1982.

[4] K. Dajani and A. Fieldsteel. "Equipartition of Interval Partitions and an Application to Number Theory." *Proc. Amer. Math. Soc.* **129** (2001): 3453-3460.

[5] K. Dajani and C. Kraaikamp. "A Gauss-Kusmin Theorem for Optimal Continued Fractions." *Trans. Amer. Math. Soc.* **351** (1999): 2055-2079.

[6] K. Dajani and C. Kraaikamp. *Ergodic Theory and Numbers.* MAA, Washington, 2002.

[7] M. Embree and L.N. Trefethen. "Growth and Decay of Random Fibonacci Sequences." *Proc. Roy. Soc. Lond.* **A 455** (1999): 2471-2485.

[8] S. Finch. *Mathematical Constants.* Cambridge University Press, New York, 2003.

[9] M. Iosifescu and S. Grigorescu. *Dependence with Complete Connections and its Applications.* Cambridge University Press, New York, 1990.

[10] M. Iosifescu and C. Kraaikamp. *Metrical Theory of Continued Fractions.* Kluwer Academic Publishers, Dordrecht, 2002.

[11] V.G. Lochs. "Vergleich der Genauigkeit von Dezimalbruch und Kettenbruch." *Abh. Math. Sem. Hamburg* **27** (1964): 142-144.

[12] D. Mayer. *Continued Fractions and Related Transformations.* "Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces," Ed. Tim Bedford et al., Oxford, 1991.

[13] H. Nakada. "On the Invariant Measures for the Transformations Associated with some Real Continued-Fractions." *Keio Eng. Rep.* **30** (1977): 159-175.

[14] I. Peterson. *Uncovering a New Mathematical Constant.* Science News Online, **155** (1999), available at http://sciencenews.org/sn_arc99/6_12_99/bob1.htm.

[15] M. Pollicott and M. Yuri. *Dynamical Systems and Ergodic Theory.* Cambridge University Press, New York, 1998.

[16] V. A. Rohlin. "Exact Endomorphisms of a Lebesgue Space." *Amer. Math. Soc. Transl. II. Ser.* **39** (1964): 1-36.

[17] T. A. Schmidt. *Remarks on the Rosen λ-Continued Fractions.* "Number Theory with an Emphasis on the Markoff Spectrum," Lecture Notes in Pure and Applied Mathematics 147, Ed. A. D. Pollington and W. Moran, Oxford, 1991. Marcel Dekker, New York/Basel/Hong Kong, 1993, 227-238.

[18] F. Schweiger. *Ergodic Theory of Fibred Systems and Metric Number Theory.* Oxford University Press, Oxford, 1995.

[19] D. Viswanath. "Random Fibonacci Sequences and the Number 1.13198824...." *Math. Comput.* **69** (2000): 1131-1155.

[20] E. Wirsing. "On the Theorem of Gauss-Kusmin-Lévy and a Frobenius-type Theorem for Function Space." *Acta Arithmetica* **24** (1974): 507-528.

[21] T. G. Wright and L. N. Trefethen. "Computing Lyapunov Constants for Random Recurrences with Smooth Coefficients." *J. Comp. Appl. Math.* **132** (2001): 331-340.

AMS Classification Numbers: 11B39, 37A05

✠ ✠ ✠