A q-ANALOGUE OF GENERALIZED STIRLING NUMBERS

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ABSTRACT

We investigate a kind of q-analogue which involves a unified generalization of Stirling numbers as a limiting case with q→1. Some basic properties and explicit formulas will be derived, and certain applications related to previously known results will be discussed.

1. INTRODUCTION

It is known that some previous generalizations of Stirling numbers could be unified by starting with transformations between generalized factorials involving three arbitrary parameters. See the recent paper by Hsu and Shiue [15]; and for some of their combinatorial and statistical applications refer to Corcino, Hsu, and Tan [9].

Let us denote $(t|\alpha)_n = t(t-\alpha)...(t-n\alpha+\alpha)$ for $n = 1, 2, ...$, and define $(t|\alpha)_0 = 1$. In particular we write $(t|1)_n = (t)_n$ with $(t)_0 = 1$. The so-called unified generalization of Stirling numbers is a Stirling-type pair $\left\{S^1, S^2\right\} = \left\{S^1(n,k), S^2(n,k)\right\} \equiv \{S(n,k;\alpha,\beta,\gamma), S(n,k;\beta,\alpha,-\gamma)\}$ defined by the inverse relations (c.f. [15])

$$(t|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-\gamma|\beta)_k \tag{1}$$

$$(t|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+\gamma|\alpha)_k \tag{2}$$

where $n \in N$ (the set of nonnegative integers), α, β, and γ are real or complex numbers with $(\alpha,\beta,\gamma) \neq (0,0,0)$.

Using Knuth's convenient notations, we see that the classical Stirling numbers of the first and second kind as well as the binomial coefficient $\binom{n}{k}$ are given, respectively, by

$$S(n,k;1,0,0) = \begin{bmatrix} n \\ k \end{bmatrix}, \quad S(n,k;0,1,0) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, \quad S(n,k;0,0,1) = \binom{n}{k}.$$

Moreover, as is easily observed, various well-known extensions of Stirling numbers due to Broder[1], Carlitz [2] [3], Charalambides and Koutras [5], Gould and Hopper [13], Koutras

[16], Howard [14], Riordan [17], and others can be written as $S(n,k;\alpha,\beta,\gamma)$ with suitable choices of the parameters α, β, and γ (c.f. [15]). However, the weighted Stirling pair of Yu [19], denoted by $\{S(n,k,\alpha,\beta;t), S(n,k,\beta,\alpha;-t)\}$, and the one used in the paper-the Stirling-type pair $\{S^1(n,k), S^2(n,k)\}$ of Hsu and Shiue [15], are exactly the same pair of numbers but are defined in different ways. The former is defined by means of vertical generating functions

$$(1+\alpha x)^{t/\alpha}\left[\frac{(1+\alpha x)^{\beta/\alpha}-1}{\beta}\right]^k = k!\sum_{n=0}^{\infty} S(n,k,\alpha,\beta;t)\frac{x^n}{n!}$$

$$\frac{1}{(1+\alpha x)^{t/\alpha}}\left[\frac{(1+\beta x)^{\alpha/\beta}-1}{\alpha}\right]^k = k!\sum_{n=0}^{\infty} S(n,k,\beta,\alpha;-t)\frac{x^n}{n!}$$

while the latter is defined by means of the inverse relations (1) and (2). On the other hand, the multiparameter noncentral Stirling numbers of the first and second kinds, denoted by $s(n,k;\bar{\alpha})$ and $S(n,k;\bar{\alpha})$, respectively, which are defined by B.S. El-Desouky [11] as follows

$$(t)_n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(t/\alpha)_k$$

$$(t/\alpha)_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k$$

where $\bar{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ and

$$(t/\alpha)_n = \prod_{i=0}^{n-1}(t-\alpha_i)$$

are related with the Stirling-type pair of Hsu and Shiue in the following manner

$$S(n,k;1,\alpha,0) = s(n,k;\bar{\alpha})$$
$$S(n,k;\alpha,1,0) = S(n,k;\bar{\alpha})$$

where $\bar{\alpha} = (0.\alpha, 1.\alpha, ..., (n-1)\alpha)$.

Denote $[\alpha] = (q^\alpha - 1)/(q-1)$ with $q \neq 1$. As a q-analogue of $(t|\alpha)_n$ we may consider a factorial of t of the form

$$t(t-[\alpha])(t-[2\alpha])...(t-[(n-1)\alpha]), \quad n \geq 1.$$

Multiplying this with $(q-1)^n$, we get the product

$$(x-q^0)(x-q^\alpha)...(x-q^{(n-1)\alpha})$$

where $x = t(q-1)+1$. This suggests the definition

$$[t|q^\alpha]_n = \prod_{j=0}^{n-1}(t - q^{j\alpha}), \quad n \ge 1, \quad [t|q^\alpha]_0 = 1.$$

For simplicity we denote $a = q^\alpha$ with $a \neq 1$, so that $[t|a]_n$ takes the form

$$[t|a]_n = \prod_{j=0}^{n-1}(t - a^j), \quad [t|a]_0 = 1, \quad [t|a]_1 = t-1.$$

This may be called *exponential factorial of t with base a.*

Accordingly, parallel to (1) and (2) we may introduce a new kind of generalized Stirling number-pair $\{S^1[n,k], S^2[n,k]\}$ by the inverse relations

$$[t|a]_n = \sum_{k=0}^{n} S^1[n,k][t-c|b]_k \tag{3}$$

$$[t|b]_n = \sum_{k=0}^{n} S^2[n,k][t+c|a]_k \tag{4}$$

where a, b, and c are real or complex parameters with $a \neq 1$, $b \neq 1$, and $S^1[n,k]$ and $S^2[n,k]$ may be denoted more precisely as $S^1[n,k] = S[n,k;a,b,c]$ and $S^2[n,k] = S[n,k;b,a,-c]$.

We also define the value $S^1[n,k] = S^2[n,k] = 0$ for $k > n$. Generally, $\{S^1[n,k], S^2[n,k]\}$ may be called a *pair of exponential-type Stirling numbers.* We will investigate some basic properties of such a pair of numbers in subsequent sections. In particular, it will be shown (in section 4) that the following limit relations hold:

$$\lim_{q\to 1} S[n,k;q^\alpha,q^\beta,q^\gamma-1](q-1)^{k-n} = S(n,k;\alpha,\beta,\gamma) \tag{5}$$

$$\lim_{q\to 1} S[n,k;q^\beta,q^\alpha,1-q^\gamma](q-1)^{k-n} = S(n,k;\beta,\alpha,-\gamma). \tag{6}$$

This indicates that the exponential-type Stirling numbers could be regarded as a q-analogue for the generalized Stirling numbers $S^1(n,k)$ and $S^2(n,k)$.

2. ORTHOGONALITY AND RECURRENCE RELATIONS

Evidently the sequences $[t|a]_n$ and $[t-c|b]_n$ with $(b,c) \neq (a,0)$ provide two different sets of bases for the linear space of polynomials in t, so that by substituting (3) into (4) (or (4) into (3)), we may easily obtain the orthogonality relations stated in the next proposition.

Proposition 1: The numbers $S^1[n,k]$ and $S^2[n,k]$ satisfy the orthogonality relations

$$\sum_{k=n}^{m} S^1[m,k]S^2[k,n] = \sum_{k=n}^{m} S^2[m,k]S^1[k,n] = \delta_{mn} \quad (m \ge n) \tag{7}$$

where δ_{mn} is the Kronecker delta defined by $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ if $m \neq n$.

As a consequence of (7) we have the inverse relations for $n \in N$:

$$f_n = \sum_{k=0}^{n} S^1[n,k] g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} S^2[n,k] f_k \tag{8}$$

$$f_n = \sum_{k=n}^{\infty} S^1[k,n] g_k \Longleftrightarrow g_n = \sum_{k=n}^{\infty} S^2[k,n] f_k \tag{9}$$

where the sequences $\{f_k\}$ and $\{g_k\}$ are assumed to be an ultimately vanishing sequence.

Hereafter, it suffices to consider $S^1[n,k] \equiv S[n,k;a,b,c]$ since a, b, and c are arbitrary parameters. Note that (3) is a linear relation involving polynomials in t with highest degree n in t, so that (3) implies

$$S^1[n,n] = 1 \quad (n \geq 0), \qquad S^1[1,0] = c.$$

Proposition 2: The numbers $S^1[n,k]$ satisfy the recurrence relations

$$S^1[n+1,k] = S^1[n,k-1] + (b^k - a^n + c) S^1[n,k] \tag{10}$$

where $n \geq k \geq 1$ and $S^1[n,0] = [1+c|a]_n$.

Proof: Taking $t = c+1$ in (3) we have

$$[c+1|a]_n = \sum_{k=0}^{n} S^1[n,k][1|b]_k = S^1[n,0].$$

To prove (10) we may start with (3) and proceed as follows

$$\begin{aligned}
\sum_{k=0}^{n+1} S^1[n+1,k][t-c|b]_k &= [t|a]_{n+1} = [t|a]_n (t - a^n) \\
&= \sum_{k=0}^{n} S^1[n,k][t-c|b]_{k+1} \\
&\qquad + \sum_{k=0}^{n} S^1[n,k](b^k - a^n + c)[t-c|b]_k \\
&= \sum_{k=1}^{n+1} S^1[n,k-1][t-c|b]_k \\
&\qquad + \sum_{k=0}^{n} S^1[n,k](b^k - a^n + c)[t-c|b]_k.
\end{aligned}$$

Thus (10) follows from the comparison of the coefficients of $[t-c|b]_k$ in the first and last expressions. $\square$

It may be worth noticing that (10) represents a new kind of linear partial difference equations, and the exponential-type Stirling numbers just provide a solution to such kind of equations with certain given initial conditions.

Proposition 3: For given $n \geq k \geq 0$ there hold the horizontal recurrence relations with $b \neq 1$:

$$[b^k|b]_k S^1[n,k] = [b^k + c|a]_n - \sum_{j=0}^{k-1} S^1[n,j][b^k|b]_j \tag{11}$$

where the case for $k = 0$ is given by $S^1[n,0] = [1 + c|a]_n$.

Proof: It is clear that $[b^k|b]_j = 0$ for $j \geq k+1$. Thus by taking $t = b^k + c$ in (3) we immediately obtain

$$[b^k + c|a]_n = \sum_{j=0}^{n} S^1[n,j][b^k|b]_j = \sum_{j=0}^{k} S^1[n,j][b^k|b]_j \tag{12}$$

which is precisely (11). □

Example: For $k = 1$ and 2, (11) gives

$$S^1[n,1] = \left([b+c|a]_n - [1+c|a]_n\right)/(b-1)$$

$$S^1[n,2] = \left(\frac{1}{b}[b^2+c|a]_n - \frac{b+1}{b}[b+c|a]_n + [1+c|a]_n\right)/(b-1)(b^2-1).$$

3. EXPLICIT FORMULAS

We will find explicit expressions for both $S^1(n,k)$ and $S^1[n,k]$. These expressions are analogous to each other and can be used to prove the limit relations (5) and (6).

Proposition 4: Let $\beta \neq 0$. Then the numbers $S^1(n,k) = S(n,k;\alpha,\beta,\gamma)$ defined by (1) can be written in the form

$$S^1(n,k) = \frac{1}{k!\beta^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + \gamma|\alpha)_n \tag{13}$$

where $n \geq k \geq 0$.

Proof: Replacing t with $\beta t + \gamma$, we see that the basic equation (1) takes the form

$$(\beta t + \gamma|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(\beta t|\beta)_k$$

$$= \sum_{k=0}^{n} S^1(n,k)\beta^k k! \binom{t}{k}.$$

It follows from Newton's interpolation formula

$$k!\beta^k S^1(n,k) = \Big[\Delta^k(\beta t+\gamma|\alpha)_n\Big]_{t=0} \tag{14}$$

where Δ is the difference operator defined by $\Delta f(t) = f(t+1) - f(t)$. Clearly, (14) may be rewritten more explicitly in the form (13). $\square$

Remark: Comparing (14) (with $\alpha = 1$) with the definition of the well-known C-numbers first thoroughly investigated by Charalambides and Koutras [5] and also by Howard [14], one may find that $S(n,k;1,\beta,\gamma)$ are essentially equivalent to C-numbers.

In what follows we will make use of the Gaussian polynomial or q-binomial coefficient (with $q \neq 1$) defined by

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \prod_{i=1}^{j} \frac{q^{k-i+1}-1}{q^i-1}, \qquad \begin{bmatrix} k \\ 0 \end{bmatrix}_q = 1,$$

and also the q-binomial inversion formula for $k \in N$:

$$f_k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q g_j \Longleftrightarrow g_k = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f_j. \tag{15}$$

In dealing with the numbers $S^1[n,k]$ we often use b instead of q. Also for the sake of simplicity we adopt the notation

$$< k|j >= \binom{k-j}{2} - \binom{k}{2} = \binom{j+1}{2} - kj.$$

Proposition 5: Let $b \neq 1$. Then the numbers $S^1[n,k] = S[n,k;a,b,c]$ defined by (3) can be expressed in the form

$$S^1[n,k] = \prod_{i=1}^{k}(b^i-1)^{-1} \sum_{j=0}^{k} (-1)^{k-j} b^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_b [b^j + c|a]_n \tag{16}$$

Proof: Observe that the factor $[b^k|b]_j$ contained in the RHS of (12) may be written in the form

$$[b^k|b]_j = b^{\binom{j}{2}}(b^k-1)(b^{k-1}-1)...(b^{k-j+1}-1)$$

$$= b^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_b \prod_{i=1}^{j}(b^i-1).$$

Consequently, (12) may be expressed as a q-binomial transform

$$[b^k + c|a]_n = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_b S^1[n,k] b^{\binom{j}{2}} \prod_{i=1}^{j}(b^i-1) \tag{17}$$

where n may be regarded as a fixed parameter.

Using the inversion formulae (15), we can invert (17):

$$b^{\binom{k}{2}} \prod_{i=1}^{k}(b^i - 1) \cdot S^1[n,k] = \sum_{j=0}^{k}(-1)^{k-j} b^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_b [b^j + c|a]_n,$$

which is precisely (16). $\square$

Evidently, (16) is an analogue of (13) in which the binomial coefficients and generalized factorials are replaced by the q-binomial coefficients and exponential factorials, respectively. However, the close relationship between this pair of explicit formulas can only be disclosed with the substitution $a = q^\alpha$, $b = q^\beta$, and $c = q^\gamma - 1$. This will be shown in the next section.

4. A KIND OF q-ANALOGUE

Let $a = q^\alpha$, $b = q^\beta$, and $c = q^\gamma - 1$ with $q \neq 1$. Then a pair of q-Stirling numbers $\sigma^1[n,k]$ and $\sigma^2[n,k]$ may be introduced via $S^1[n,k]$ and $S^2[n,k]$. More precisely we have the following

Definition: For $\alpha \neq 0$ and $\beta \neq 0$ we define

$$\begin{aligned} \sigma^1[n,k] &\equiv \sigma^1[n,k;\alpha,\beta,\gamma]_q := S[n,k;q^\alpha,q^\beta,q^\gamma - 1](q-1)^{k-n} \\ \sigma^2[n,k] &\equiv \sigma^2[n,k;\alpha,\beta,\gamma]_q := S[n,k;q^\beta,q^\alpha,1 - q^\gamma](q-1)^{k-n}, \end{aligned}$$

where $\sigma^1[0,0] = \sigma^2[0,0] = 1$, and the case $\alpha = 0$ or $\beta = 0$ is treated as the limit as $\alpha \to 0$ or $\beta \to 0$ whenever the limit exists.

Proposition 6: There hold the limit relations

$$\lim_{q\to 1} \sigma^1[n,k;\alpha,\beta,\gamma]_q = S(n,k;\alpha,\beta,\gamma) \qquad (5 \text{ bis})$$

$$\lim_{q\to 1} \sigma^2[n,k;\alpha,\beta,\gamma]_q = S(n,k;\beta,\alpha,-\gamma) \qquad (6 \text{ bis})$$

where the parameter β contained in (5 bis) and α in (6 bis) are assumed to be different from zero.

Proof: It suffices to prove (5 bis) since the case for (6 bis) can be treated similarly. Making use of Proposition 5 with $a = q^\alpha$, $b = q^\beta$, and $c = q^\gamma - 1$, and adopting the notation $[x] = (q^x - 1)/(q - 1)$, we may write, in accordance with the definition,

$$\sigma^1[n,k] = \left(\prod_{i=1}^{k}[i\beta]\right)^{-1} \sum_{j=0}^{k}(-1)^{k-j} b^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_b \frac{[q^{\beta j} + q^\gamma - 1|q^\alpha]_n}{(q-1)^n},$$

where $\prod_{i=1}^{k}[i\beta] = 1$ for $k = 0$. Clearly we have

$$\lim_{q\to 1}\left(\prod_{i=1}^{k}[i\beta]\right)^{-1} = \frac{1}{k!\beta^k}$$

$$\lim_{q\to 1} b^{<k|j>}\begin{bmatrix}k\\j\end{bmatrix}_b = \binom{k}{j}$$

$$\lim_{q\to 1}\frac{[q^{\beta j}+q^{\gamma}-1|q^{\alpha}]_n}{(q-1)^n} = \lim_{q\to 1}\prod_{i=0}^{n-1}([\beta j]+[\gamma]-[\alpha i])$$

$$= \prod_{i=0}^{n-1}(\beta j+\gamma-\alpha i) = (\beta j+\gamma|\alpha)_n.$$

Consequently we get with the aid of Proposition 4

$$\lim_{q\to 1}\sigma^1[n,k] = \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma|\alpha)_n = S^1(n,k).$$

This completes the proof of (5 bis). □

Using Proposition 1 and the definition of $\sigma^1[n,k]$ and $\sigma^2[n,k]$, the following proposition can be easily verified.

Proposition 7: The numbers $\sigma^1[n,k]$ and $\sigma^2[n,k]$ satisfy the orthogonality relations

$$\sum_{k=n}^{m}\sigma^1[m,k]\sigma^2[k,n] = \sum_{k=n}^{m}\sigma^2[m,k]\sigma^1[k,n] = \delta_{mn}. \tag{18}$$

And consequently there hold the inverse relations

$$f_n = \sum_{k=0}^{n}\sigma^1[n,k]g_k \Longleftrightarrow g_n = \sum_{k=0}^{n}\sigma^2[n,k]f_k \tag{19}$$

$$f_n = \sum_{k=n}^{\infty}\sigma^1[k,n]g_k \Longleftrightarrow g_n = \sum_{k=n}^{\infty}\sigma^2[k,n]f_k \tag{20}$$

where the sequences $\{f_k\}$ and $\{g_k\}$ are assumed to be ultimately vanishing.

Parallel to Proposition 2 we have

Proposition 8: The number $\sigma^1[n,k] = \sigma^1[n,k;\alpha,\beta,\gamma]_q$ satisfy the recurrence relations

$$\sigma^1[n+1,k] = \sigma^1[n,k-1] + ([k\beta]-[n\alpha]-[\gamma])\,\sigma^1[n,k] \tag{21}$$

where $n \geq k \geq 1$, $\sigma^1[n,n] = \sigma^2[0,0] = 1$ and $\sigma^1[n,0] = [q^{\gamma}|q^{\alpha}]_n(q-1)^{-n}$.

Proof: Equation (21) can be easily deduced from the definition of $\sigma^1[n,k]$ and from (10) with $a = q^{\alpha}$, $b = q^{\beta}$, and $c = q^{\gamma}-1$, by noticing that $(b^k-a^n+c)/(q-1) = [(q^{k\beta}-1)-(q^{n\alpha}-1)+$

$(q^\gamma - 1)]/(q-1) = [k\beta] - [n\alpha] + [\gamma]$, and that $\sigma^1[n,0] = [1+c|a]_n(q-1)^{-n} = [q^\gamma|q^\alpha]_n(q-1)^{-n}$. $\square$

As $q \to 1$ we see that the limiting form of (21) yields

$$S(n+1,k) = S(n,k-1) + (k\beta - n\alpha + \gamma)S(n,k). \tag{22}$$

This is the same basic recurrence relation for the generalized Stirling numbers, with $S^1(n,0) = (\gamma|\alpha)_n$. (c.f. theorem 1 of [15]). Certainly, the limiting forms of (19)-(20) also yield the inversion formulas involving the pair of generalized Stirling numbers.

5. SOME REMARKS AND RELATED PROBLEMS

Here we will state several remarks containing some open problems related to q-analogues of generalized Stirling numbers. These problems are mainly suggested by Propositions 4 and 5, and the similarity between them.

Remark 1: The explicit formula (13) of Proposition 4 can be used to provide a simpler and more straightforward proof for the generating function of numbers $S^1(n,k)$:

$$\frac{1}{k!}(1+\alpha t)^{\gamma/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta}\right)^k = \sum_{n=0}^{\infty} S^1(n,k)\frac{t^n}{n!} \tag{23}$$

where $\alpha\beta \neq 0$. Note that (23) was proved in Théorêt's paper [18] using two lemmas, and was also proved in [15] using difference-differential equations and somewhat tedious computations.

Let us rewrite (13) in the form

$$S^1(n,k) = \frac{n!\alpha^n}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j + (\gamma/\alpha)}{n}. \tag{24}$$

Proof of (23): Starting with (24) and making use of Vandermonde's convolution identity and Cauchy's rule for the multiplication of series, we find

$$\text{The RHS of (23)} = \sum_{n=0}^{\infty}\frac{\alpha^n}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j + (\gamma/\alpha)}{n}t^n$$

$$= \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{n=0}^{\infty}(\alpha t)^n\sum_{\lambda=0}^{n}\binom{\gamma/\alpha}{\lambda}\binom{(\beta/\alpha)j}{n-\lambda}\right\}$$

$$= \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{\lambda=0}^{\infty}\binom{\gamma/\alpha}{\lambda}(\alpha t)^\lambda\sum_{\mu=0}^{\infty}\binom{(\beta/\alpha)j}{\mu}(\alpha t)^\mu\right\}$$

$$= \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1+\alpha t)^{\gamma/\alpha}(1+\alpha t)^{(\beta/\alpha)j}$$

$$= \frac{1}{k!\beta^k}(1+\alpha t)^{\gamma/\alpha}\left[(1+\alpha t)^{\beta/\alpha} - 1\right]^k = \text{The LHS of (23)}.$$

Hence (23) is proved. $\square$

The above proof suggest the following problem.

Problem 1: Is it possible to start with the explicit formula (16) of Proposition 5 to construct a generating function for $S^1[n,k]$ as well as for $\sigma^1[n,k]$?

Remark 2: The following vertical recurrence relation

$$kS^1(n,k) = \frac{1}{\beta}\sum_{j=k-1}^{n-1}\binom{n}{j}(\beta|\alpha)_{n-j}S^1(j,k-1) \tag{25}$$

is obtained by Corcino [8]. A simple proof of this relation may be obtained straightforwardly from (13). Since $S^1(j-k-1) = 0$ for $j < k-1$, we now rewrite (25) in the form

$$k!\beta^k S^1(n,k) = \sum_{j=0}^{n-1}\binom{n}{j}(\beta|\alpha)_{n-j}S^1(j,k-1)(k-1)!\beta^{k-1}. \tag{26}$$

Proof of (26): Using (13) with the aid of Vandermonde's convolution identity, we see that

$$\text{The RHS of (26)} = \sum_{j=0}^{n-1}\binom{n}{j}(\beta|\alpha)_{n-j}\sum_{\lambda=0}^{k-1}(-1)^{k-\lambda-1}\binom{k-1}{\lambda}(\beta\lambda+\gamma|\alpha)_j$$

$$\begin{aligned}
&= \sum_{\lambda=0}^{k-1}(-1)^{k-\lambda-1}\binom{k-1}{\lambda}\sum_{j=0}^{n-1}\binom{n}{j}(\beta|\alpha)_{n-j}(\beta\lambda+\gamma|\alpha)_j \\
&= \sum_{\lambda=0}^{k-1}(-1)^{k-\lambda-1}\binom{k-1}{\lambda}[(\beta+\beta\lambda+\gamma|\alpha)_n - (\beta\lambda+\gamma|\alpha)_n] \\
&= \sum_{\lambda=0}^{k-1}(-1)^{k-\lambda}\binom{k-1}{\lambda}(\beta\lambda+\gamma|\alpha)_n + \sum_{\lambda=0}^{k}(-1)^{k-\lambda}\binom{k-1}{\lambda-1}(\beta\lambda+\gamma|\alpha)_n \\
&= \sum_{\lambda=0}^{k}(-1)^{k-\lambda}\binom{k}{\lambda}(\beta\lambda+\gamma|\alpha)_n = \text{The LHS of (26)}.
\end{aligned}$$

This also suggests a related problem for q-Stirling numbers.

Problem 2: How do we make use of the formula (16) to establish a vertical recurrence relation for $S^1[n,k]$ and $\sigma^1[n,k]$?

We believe that this problem may be much easier to solve than Problem 1.

Remark 3: As is shown in the proof of Proposition 6, the numbers $\sigma^1[n,k]$ could be expressed more explicitly in the form

$$\sigma^1[n,k] = \sum_{j=0}^{k}(-1)^{k-j}q^{\beta<k|j>}\begin{bmatrix}k\\j\end{bmatrix}_b\frac{\prod_{i=0}^{n-1}([\beta j]+[\gamma]-[\alpha i])}{\prod_{i=1}^{k}[\beta i]} \tag{27}$$

where $< k|j >= \binom{j+1}{2} - kj$ and $b = q^{\beta}$. This implies some interesting cases for some special values of α, β, and γ.

For instance, taking $(\alpha, \beta, \gamma) = (\theta, 1, 0)$, we may obtain a q-analogue of Carlitz's degenerate Stirling numbers [2]:

$$\sigma[n, k; \theta, 1, 0]_q = \frac{1}{[k]!} \sum_{j=0}^{k} (-1)^{k-j} q^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_q \prod_{i=0}^{n-1} ([j] - [\theta i]) \tag{28}$$

where $[k]! = [1][2]...[k]$. In particular, for $\theta = 0$ we may get the q-Stirling numbers of the second kind due to Carlitz [4]:

$$\sigma[n, k; 0, 1, 0]_q = \frac{1}{[k]!} \sum_{j=0}^{k} (-1)^{k-j} q^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]^n. \tag{29}$$

This was discussed more thoroughly in Gould's paper [12]. Obviously (29) implies the explicit form of the second kind of Stirling numbers as a limiting case:

$$\lim_{q \to 1} \sigma[n, k; 0, 1, 0]_q = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n = S(n, k; 0, 1, 0). \tag{30}$$

Remark 4: Note that the general expression (27) does not imply any available formula for the q-analogue of the first kind Stirling numbers $S(n, k; 1, 0, 0)$, inasmuch as $\beta = 0$ will present zero factors in the denominators of fractions appearing in (27). It is known that, in the classical case, the explicit form for the first kind of Stirling numbers is given by Schlömilch's formula which expresses a linear relation between the two kinds of Stirling numbers (c.f. [7]). Thus, we may propose

Problem 3: How do we make use of the basic relations (3) and (4) together with (16) to find certain linear relations between the two kinds of numbers, $S^1[n, k]$ and $S^2[n, k]$, or $\sigma^1[n, k]$ and $\sigma^2[n, k]$, so that a generalized Schlömilch formula could be established?

Surely, still much remains to be done, and any solutions to the above-mentioned problems would substantially enrich the theory of q-analogues of generalized Stirling numbers. See also an approach via the combinatorial study of 0-1 tableau by de Medicis and Leroux [10].

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