GENERALIZED MÖBIUS INVERSION - THEORETICAL AND COMPUTATIONAL ASPECTS

P. Bundschuh
Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

L. C. Hsu
Math. Institute, Dalian Institute of Technology, Dalian, P. R. China

P. J.-S. Shiue
Dept. of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada

(Submitted October 2003-Final Revision April 2004)

ABSTRACT

We consider the problem for finding a wide class of Möbius-type inversion formulae via Dirichlet series representations. We provide theoretical and computational aspects of this problem. Several examples are given to illustrate the constructive method.

1. INTRODUCTION

This paper is concerned with the problem of construction for a general type of Möbius inversion formulae in number theory and combinatorial theory.

We use $\mathbb{N}$ and $\mathbb{C}$ to denote the set of positive integers and of complex numbers, respectively, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Usually a number-theoretical function F (i.e. a map $F : \mathbb{N} \to \mathbb{C}$) is called *multiplicative* if

$$F(mn) = F(m)F(n) \tag{1}$$

holds whenever $m, n \in \mathbb{N}$ are relatively prime. This definition of multiplicity implies either $F(1) = 0$ (and thus $F = 0$ identically) or $F(1) = 1$. If M denotes the set of all multiplicative number-theoretical functions, then it is well-known that $M - \{0\}$ forms an abelian group with respect to the (Dirichlet) convolution $*$, the unit element ε being defined by $\varepsilon(1) := 1$ and $\varepsilon(n) := 0$ for any $n > 1$ (compare, e.g. [2, Satz 1.4.8]).

To exclude $F = 0$ from the set of multiplicative functions several authors use a more narrow definition of multiplicity by requiring (1) *and* $F(1) = 1$.

Next we recall a fairly more general notion of multiplicativity, introduced first by Selberg [8], which apparently did not prevail in the literature. A number-theoretical function F is said to be *Selberg-multiplicative* if, for each prime p, there exists $f_p : \mathbb{N}_0 \to \mathbb{C}$ with $f_p(0) = 1$ for all but finitely many p such that

$$F(n) = \prod_p f_p\left(e_p(n)\right)$$

holds for every $n \in \mathbb{N}$, where $e_p(n)$ denotes the exact exponent of p in the canonical factorization of n. One of the main advantages of this more general notion of multiplicativity is that it can be used without change to define multiplicative functions of several variables.

We consider now the particular subclass S of Selberg-multiplicative $F : \mathbb{N} \to \mathbb{C}$ such that there exists a universal $f : \mathbb{N}_0 \to \mathbb{C}$ with $f(0) = 1$ such that, for any $n \in \mathbb{N}$,

$$F(n) = \prod_p f\left(e_p(n)\right) \tag{2}$$

Clearly, every $F \in S$ is multiplicative in the "usual" sense mentioned at the beginning, and satisfies $F(1) = 1$. This means $S \subset M - \{0\}$. We note that Carlitz [3] and Knopfmacher [7] have examined a similar set of functions.

Obviously, the classical Möbius function μ belongs to S as well as certain of its generalizations, e.g. Fleck's [6] μ_z, which, for any $z \in \mathbb{C}$, is defined by

$$\mu_z(n) := \prod_p (-1)^{e_p(n)} \cdot \binom{z}{e_p(n)} \tag{3}$$

for any $n \in \mathbb{N}$. Clearly, $\mu_0 = \varepsilon$ and $\mu_1 = \mu$. (Note already here that $\mu_{-1} = 1$, the function which is identically 1, and $\mu_{-2} = \tau$, the classical divisor function.) This motivates us to investigate the problem for the construction of a general type of Möbius inversion as follows.

Problem: *Given $F \in S$, can we find $G \in S$ such that there holds a Möbius-type inversion of the form*

$$\beta(n) = \sum_{d|n} F\left(\frac{n}{d}\right) \alpha(d) \Leftrightarrow \alpha(n) = \sum_{d|n} G\left(\frac{n}{d}\right) \beta(d) \tag{4}$$

for number-theoretical functions α, β, and the summations are both taken over all the positive divisors d of n?

The objective of this paper is to solve this problem first theoretically (in section 2), and then constructively and explicitly (in section 3). It will be shown that G is uniquely determined by the given F. Then $\{F, G\}$ may be called a *reciprocal pair of generalized Möbius functions.*

2. GROUP-THEORETICAL ASPECT OF THE CLASS S

Writing $\check{F}$ for the inverse of $F \in M - \{0\}$ with respect to convolution $*$ it will be clear that our above Problem has a unique solution if we can show that $F \in S$ implies $\check{F} \in S$. Namely we can write (4) equivalently as $\beta = F * \alpha \Leftrightarrow \alpha = G * \beta$, and so we have just to take $G = \check{F}$.

The following Proposition gives even a little more information than needed.

Proposition: *$\langle S, * \rangle$ is a subgroup of $\langle M - \{0\}, * \rangle$.*

Proof: For this we have to show $F, G \in S \Rightarrow H = F * G, \check{F} \in S$. Let $f, g : \mathbb{N}_0 \to \mathbb{C}$ with $f(0) = 1, g(0) = 1$ be such that (2) and

$$G(n) = \prod_p g\left(e_p(n)\right) \tag{5}$$

hold for any $n \in \mathbb{N}$. Defining $h : \mathbb{N}_0 \to \mathbb{C}$ pointwise by

$$h(r) := \sum_{\rho=0}^{r} f(\rho) g(r - \rho) \quad (r \in \mathbb{N}_0) \tag{6}$$

we see $h(0) = 1$ and furthermore, since $H = F * G \in M$,

$$H(n) = \prod_p H\left(p^{e_p(n)}\right) = \prod_p \sum_{\rho=0}^{e_p(n)} F\left(p^\rho\right) G\left(p^{e_p(n)-\rho}\right)$$

$$= \prod_p \sum_{\rho=0}^{e_p(n)} f(\rho) g\left(e_p(n) - \rho\right) = \prod_p h\left(e_p(n)\right)$$

where we used (2), (5) and (6), respectively, to get the last two equalities.

To prove $F \in S \Rightarrow \check{F} \in S$ we define $\Psi : \mathbb{N}_0 \to \mathbb{C}$ again pointwise by

$$\Psi(0) = 1, \sum_{\rho=0}^{r} f(\rho)\Psi(r-\rho) = 0 \quad (r = 1, 2, \cdots); \tag{7}$$

since $f(0) = 1$ this is uniquely possible. From $F * \check{F} = \varepsilon$ and (2) we get

$$0 = \varepsilon\left(p^r\right) = \prod_{\rho=0}^{r} F(p^\rho)\check{F}(p^{r-\rho}) = \sum_{\rho=0}^{r} f(\rho)\check{F}(p^{r-\rho})$$

for each prime p and $r \in \mathbb{N}$. Then we see from (7) that $\check{F}(1) = 1 = \Psi(0)$, and inductively $\check{F}(p^r) = \Psi(r)$ for each prime p and $r \in \mathbb{N}$. Thus we have

$$\check{F}(n) = \prod_p \check{F}\left(p^{e_p(n)}\right) = \prod_p \Psi\left(e_p(n)\right),$$

whence $\check{F} \in S$.

3. METHOD OF CONSTRUCTION

Here we will make use of the following well-known proposition (see, e.g., [2, Satz 1.4.4] or [9, § 2.6]).

Lemma: *If $F \in M$, then for $s \in \mathbb{C}$ we have the formal identity*

$$\sum_{n=1}^{\infty} F(n)n^{-S} = \prod_p \sum_{r=0}^{\infty} F(p^r)p^{-rs}.$$

Now suppose $F \in S$ as in our above Problem. From (2) and the Lemma we find

$$\sum_{n=1}^{\infty} F(n)n^{-s} = \prod_p \sum_{r=0}^{\infty} f(r)p^{-rs}. \tag{8}$$

Moreover, suppose that the formal power series in $x = p^{-s}$ in the right-hand side of (8) has a formal inverse power series, viz.

$$\left(\sum_{r=0}^{\infty} f(r)x^r\right)^{-1} = \sum_{r=0}^{\infty} g(r)x^r \tag{9}$$

with $g(0) = 1$ (compare $f(0) = 1$). Then a number-theoretical function G could be constructed by formula (5), i.e.,

$$G(n) := \prod_p g\left(e_p(n)\right).$$

This shows $G \in S$, so that by the Lemma we have the formal identity

$$\sum_{n=1}^{\infty} G(n)n^{-s} = \prod_p \sum_{r=0}^{\infty} g(r)p^{-rs}. \tag{10}$$

Let the left-hand side of (8) and of (10) be denoted by $\hat{f}(s)$ and $\hat{g}(s)$, respectively. Then (9) implies

$$\hat{g}(s) = 1/\hat{f}(s). \tag{11}$$

Remark 1: The formal equation (9) is obviously equivalent to $\sum_{\rho=0}^{r} f(\rho)g(r-\rho) = 0$ for $r = 1, 2, \cdots$. Comparing this with (7), and taking $g(0) = 1$ into account, we find $g = \Psi$ as expected.

Actually, relation (11) together with the familiar rule for the multiplication of generating Dirichlet series implies a general type of Möbius inversion as stated in the following Theorem, for which we gave implicitly a first proof at the beginning of section 2.

Theorem: *Suppose $F \in S$. Then $G \in S$ can be constructed via* (2), (9) *and* (5) *so that the general type of Möbius inversion* (4) *is valid. Namely*

$$\beta(n) = \sum_{d|n} F\left(\frac{n}{d}\right)\alpha(d) \Leftrightarrow \alpha(n) = \sum_{d|n} G\left(\frac{n}{d}\right)\beta(d)$$

holds for $\alpha, \beta : \mathbb{N} \to \mathbb{C}$, where either of α and β may be given arbitrarily.

Indeed, if U, V are any number-theoretical functions and if they are represented by generating Dirichlet series

$$\hat{u}(s) := \sum_{n=1}^{\infty} U(n)n^{-s}, \ \hat{v}(s) := \sum_{n=1}^{\infty} V(n)n^{-s},$$

which may be denoted in brief by

$$\hat{u}(s) \xleftrightarrow{\ Dir\ } (U(n))_{n\varepsilon\mathbb{N}}, \quad \hat{v}(s) \xleftrightarrow{\ Dir\ } (V(n))_{n\varepsilon\mathbb{N}},$$

then the well-known rule for the Dirichlet series multiplication (in fact: Dirichlet convolution) gives

$$\hat{u}(s)\hat{v}(s) \xleftrightarrow{\ Dir\ } \left(\sum_{d|n} U\left(\frac{n}{d}\right) V(d)\right)_{n\in\mathbb{N}} .$$

Consequently, by supposing that

$$\hat{a}(s) \xleftrightarrow{\ Dir\ } (\alpha(n))_{n\varepsilon\mathbb{N}}\,, \quad \hat{b}(s) \xleftrightarrow{\ Dir\ } (\beta(n))_{n\varepsilon\mathbb{N}}\,,$$

we see that the two equations (4) just correspond, respectively, to the following relations

$$\hat{b}(s) = \hat{f}(s)\hat{a}(s), \quad \hat{a}(s) = \hat{g}(s)\hat{b}(s).$$

Evidently, these two relations are equivalent to each other, in accordance with (11). Hence the theorem is proved.

Remark 2: Using the Lemma we see that for every reciprocal pair of generalized Möbius functions $\{F, G\}$ as mentioned in the Theorem, there always hold the identities (8) and (10).

Remark 3: It is clear that the process of construction for (4) involves (2), (5), and (9), in which the series expansion (9) appears to be most critical. However, in the general case, the coefficients $g(r)$ defined by (9) may be determined explicitly by means of partial Bell polynomials $B_{r,k}(1!f(1), 2!f(2), \cdots)$.

More precisely, using the well-known results to be found in Comtet [5, § 3.3], one can express $g(r), r \in \mathbb{N}$, in terms of $f(1), f(2), \cdots, f(r)$ in the explicit form

$$g(r) = \sum_{\pi(r)} (-1)^k k! \frac{f(1)^{k_1}}{k_1!} \cdot \frac{f(2)^{k_2}}{k_2!} \cdots , \tag{12}$$

where $\pi(r)$ denotes the set of partitions of r, so that the summation is taken over all the partitions $1^{k_1}2^{k_2}\cdots$ (with all $k_i \in \mathbb{N}_0$) of r satisfying the conditions

$$1 \cdot k_1 + 2 \cdot k_2 + \cdots = r, k_1 + k_2 + \cdots = k \quad (\Rightarrow 1 \le k \le r),$$

k being the number of parts of the partition $1^{k_1}2^{k_2}\cdots$. Evidently, with this remark, we conclude that the general problem of construction for (4) can be solved explicitly via (2), (9), (12) and (5).

4. EXAMPLES

In what follows we will present several examples in which both $f(r)$'s and $g(r)$'s have some simple expressions.

Example 1: Starting with (2) and taking

$$f(r) := (-1)^r \binom{z}{r} \tag{13}$$

for any $r \in \mathbb{N}_0$ and $z \in \mathbb{C}$, we have

$$\left(\sum_{r=0}^{\infty} f(r)x^r\right)^{-1} = (1-x)^{-z} = \sum_{r=0}^{\infty} \binom{-z}{r}(-x)^r$$

so that we may denote $g(r) = (-1)^r\binom{-z}{r}$ for $r \in \mathbb{N}_0$. Consequently, for each $z \in \mathbb{C}$, $\{\mu_z, \mu_{-z}\}$ is a reciprocal pair of generalized Möbius functions, compare (3). Thus the reciprocal relation (4) implies the pair of generalized Möbius inversion formulae (cf. [1])

$$\beta(n) = \sum_{d|n} \mu_z\left(\frac{n}{d}\right)\alpha(z) \Leftrightarrow \alpha(n) = \sum_{d|n} \mu_{-z}\left(\frac{n}{d}\right)\beta(d). \tag{14}$$

Evidently the classical Möbius inversion formulae are implied by (14) with $z = 1$ or $z = -1$.

Moreover, using (8) with $F = \mu_z$ and (13) we obtain

$$\sum_{n=1}^{\infty} \mu_z(n)n^{-s} = \prod_p (1-p^{-s})^z = \zeta(s)^{-z}, \tag{15}$$

where ζ is the classical Riemann zeta function. Consequently, (15) implies both of the well-known relations

$$\sum_{n=1}^{\infty} \tau(n)n^{-s} = \zeta(s)^2, \quad \sum_{n=1}^{\infty} \mu(n)n^{-s} = \zeta(s)^{-1}.$$

Note that (15) just means that

$$(\zeta(s))^{-z} \xleftrightarrow{\ Dir\ } (\mu_z(n))_{n\varepsilon\mathbb{N}}$$

for every $z \in \mathbb{C}$, and in view of our earlier investigations, this implies $\mu_z * \mu_{z'} = \mu_{z+z'}$ for any $z, z' \in \mathbb{C}$. Denoting $M_c := \{\mu_z | z \in \mathbb{C}\}$ we conclude immediately that $\langle M_c, *\rangle$ is a subgroup of our above group $\langle S, *\rangle$, compare also [1].

Example 2: The Taylor series expansions of e^x and e^{-x} may suggest us to take $f(r) := 1/r!, g(r) := (-1)^r/r!$ for any $r \in \mathbb{N}_0$. Accordingly we define

$$\lambda(n) := \prod_p \frac{1}{e_p(n)!}, \quad v(n) := \prod_p \frac{(-1)^{e_p(n)}}{e_p(n)!},$$

where $\lambda(n)$ and $v(n)$ just correspond to $F(n)$ and $G(n)$ in (2) and (5), respectively. Then, by the general inversion formula (4), we get the reciprocal relations

$$\beta(n) = \sum_{d|n} \lambda\left(\frac{n}{d}\right)\alpha(d) \Leftrightarrow \alpha(n) = \sum_{d|n} v\left(\frac{n}{d}\right)\beta(d).$$

Moreover, (8) and (10) imply the following identities

$$\sum_{n=1}^{\infty} \lambda(n)n^{-s} = \prod_p exp\left(p^{-s}\right) = exp\left(\sum_p p^{-s}\right), \sum_{n=1}^{\infty} v(n)n^{-s} = exp\left(-\sum_p p^{-s}\right).$$

Certainly these two expressions define holomorphic functions in the half-plane Re $s > 1$.

Example 3: Let $F_0 := 0, F_1 := 1$, and $F_i := F_{i-1} + F_{i-2}$ for $i \geq 2$ denote the sequence of Fibonacci numbers. Defining $f(r) := F_{r+1}$ for $r \in \mathbb{N}_0$ we see $f(0) = 1$ and

$$\left(1 - x - x^2\right)^{-1} = \sum_{r=0}^{\infty} f(r) x^r.$$

Putting $a := \left(1 + \sqrt{5}\right)/2,\ b := \left(-1 + \sqrt{5}\right)/2 = -1/a$ we have Binet's formula

$$f(r) = \frac{1}{\sqrt{5}} \left(a^{r+1} - b^{r+1}\right) \text{ for any } r \in \mathbb{N}_0.$$

Let us denote $1 - x - x^2 =: \sum_{r=0}^{\infty} g(r) x^r$, so that $g(0) = 1,\ g(1) = g(2) = -1$, and $g(r) = 0$ for $r \geq 3$.

Accordingly we define (cf. (2) and (5))

$$\xi(n) := \prod_p \left(\left(a^{1+e_p(n)} - b^{1+e_p(n)}\right)/\sqrt{5}\right), \quad \eta(n) := \prod_p g\left(e_p(n)\right).$$

Clearly, $\eta(1) = 1$ and $\eta(n) = 0$ if and only if $n \in \mathbb{N}$ is not cube-free (i.e., if there exists a prime p with $p^3 | n$), and finally $\eta(n) = (-1)^{\omega^*(n)}$ if n is cube-free and $\omega^*(n)$ denotes the number of distinct prime factors of n with $e_p(n) \in \{1, 2\}$. Consequently (4) implies the reciprocal relations

$$\beta(n) = \sum_{d|n} \xi\left(\frac{n}{d}\right) \alpha(d) \Leftrightarrow \alpha(n) = \sum_{d|n} \eta\left(\frac{n}{d}\right) \beta(d).$$

Moreover, using (8) and (10) we obtain the formal identities

$$\sum_{n=1}^{\infty} \xi(n) n^{-s} = \prod_p \left(1 - p^{-s} - p^{-2s}\right)^{-1}, \ \sum_{n=1}^{\infty} \eta(n) n^{-s} = \prod_p \left(1 - p^{-s} - p^{-2s}\right).$$

These expressions define functions holomorphic in Re $s > 1$. In particular, the second one may be compared with the famous identity $\sum_{n \geq 1} \mu(n) n^{-s} = \prod_p \left(1 - p^{-s}\right) = \zeta(s)^{-1}$.

Remark 4: With the examples given above, one may observe that even Bernoulli numbers, Euler numbers, and Catalan numbers together with their generating functions could be called into play in the construction of Möbius-type inversion formulae of the form (4). Perhaps these would serve as additional examples to justify the availability of the method expounded in section 3.

5. A FINAL REMARK

Though formula (12) gives a general expression of $g(r)$ in terms of $f(1), \cdots, f(r)$, it seems not so useful for the computation of $g(r)$ even with simpler f's. For instance, for the cases such as (13) for fixed $z \in \mathbb{C}$, the $g(r)$'s are easily determined via (9) (cf. Example 1). However, in using (12) for computing g it involves finding non-trivial relations such as

$$\sum_{\pi(r)} (-1)^k \frac{k!}{k_1!k_2!\cdots} \binom{z}{1}^{k_1} \binom{z}{2}^{k_2} \cdots = \binom{-z}{r} \tag{16}$$

or even only

$$\sum_{\pi(r)} (-1)^k \frac{k!}{k_1!k_2!\cdots} = \begin{cases} -1 \text{ when } r = 1, \\ 0 \text{ when } r \geq 2, \end{cases} \tag{17}$$

in the particular case $z = -1$.

Actually, the verification of (16) and even (17) is a bit of a tedious job, although it could be done by using Charalambides-Singh [4, (3.20), (3.24)] and Comtet [5, 3.3.6)], respectively. More precisely, (16) and (17) are equivalent to the following less familiar identities

$$\sum_{k=1}^{r} (-1)^k \frac{k!}{r!} C(r,k,z) = \binom{-z}{r}, \; \sum_{k=1}^{r} (-1)^k \frac{k!}{r!} B_{r,k}(1!, 2! \cdots) = \begin{cases} -1 \text{ when } r = 1, \\ 0 \text{ when } r \geq 2, \end{cases}$$

where $B_{r,k}$ are as in Remark 3, and $C(\dots)$ are known as C-numbers (see loc.cit.).

REFERENCES

[1] T. C. Brown, L.C. Hsu, J. Wang, and P.J.S. Shiue. "On a Certain Kind of Generalized Number-theoretical Möbius Function." *Math. Sci.* **25** (2000): 72-77.

[2] P. Bundschuh. *Einführung in die Zahlentheorie.* 5. Aufl., Springer, Berlin et. al. (2002).

[3] L. Carlitz. "Specialized Möbius Inversion." *J. Combin. Theory* Ser. A **24** (1978): 261-277.

[4] C.A. Charalambides and J. Singh. "A Review of the Stirling Numbers, Their Generalizations and Statistical Applications." *Comm. Statist. Theory Methods* **17** (1988): 2533-2595.

[5] L. Comtet. *Advanced Combinatorics.* D. Reidel, Dordrecht (1974).

[6] A. Fleck. "Über gewisse allgemeine zahlentheoretische Funktionen, insbesondere eine der Funktion $\mu(n)$ verwandte Funktion $\mu_k(m)$." *S.-B. Berlin. Math. Ges.* **15** (1916): 3-8.

[7] J. Knopfmacher. *Abstract Analytic Number Theory.* North-Holland Publishing Co., Amsterdam et. al. (1974).

[8] A. Selberg. "Remarks on Multiplicative Functions." In: Number Theory Day. *Proc. Conf., Rockefeller Univ., New York* (1976): 232-241, LNM 626, Springer, Berlin, et. al.

[9] H.S. Wilf. *Generatingfunctionology.* Academic Press, New York (1990).

AMS Classification Numbers: 11A25, 05A10, 11M41, 11B75

✠ ✠ ✠