SUMS OF THE EVEN INTEGRAL POWERS OF THE COSECANT AND SECANT

N. Gauthier
Department of Physics, The Royal Military College of Canada,
Station Forces 17 000, Kingston, ON, K7K 7B4, Canada
e-mail: gauthier-n@rmc.ca

Paul S. Bruckman
P. O. Box 150, Sointula, BC, V0N 3E0, Canada
e-mail: pbruckman@hotmail.com

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ABSTRACT

Special finite sums of the even powers of the cosecant and of the secant are studied, $\sum_k \csc^{2m}(k\pi/N)$ and $\sum_k \sec^{2m}(k\pi/N)$, with positive integers $N \geq 3, m$ and $1 \leq k < N/2$. The main result of this article is that these power sums are even polynomials in N, of order $2m$, whose coefficients are rational. The approach is based on new differential identities for the functions $\csc^2 z$ and $\sec^2 z$. The Mittag-Leffler expansions for these functions are invoked and the corresponding infinite series are summed to give closed form expressions for the desired sums. Specific polynomial coefficients are obtained, for $1 \leq m \leq 6$ and for all $N \geq 3$, to illustrate the method. Similar sums involving the cotangent and the tangent are also examined.

1. INTRODUCTION

Finding formulas for various types of finite sums constitutes an important aspect of number theory and much attention has been devoted to this subject in this journal. Applications cover combinatorial analysis, Bernoulli and Euler numbers and polynomials, number theoretic convolutions, sums of powers of integers and trigonometric sum formulas, to name just a few examples: see [2]-[13], and [16]-[18]. In this article we prove that the following sums, which involve even powers of the cosecant and secant functions, namely $\sum_k \csc^{2m}(k\pi/N)$ and $\sum_k \sec^{2m}(k\pi/N)$, with positive integers $N \geq 3$ and $1 \leq k < N/2$, are even polynomials of order $2m$ in N, with rational coefficients. The method used will also be shown to give the values of the polynomial coefficients for specific cases. We restrict the powers to be even, since corresponding sums for odd powers lead, in general, to irrational closed forms and involve more complicated analysis. As will be seen in Section IV, the even-power sums of interest here lead to rational closed forms. The main results are displayed, proofs are given and the polynomial coefficients are obtained for twelve specific cases. Our literature search indicates that the coefficients of two of the twelve given cases are known and it may well be that the other 10 are new. Closely related sums for the even powers of the cotangent and of the tangent are also presented in Section V, and shown to be polynomials also.

For N a positive integer ≥ 3, let

$$Q = \begin{cases} (N-1)/2; & N \text{ odd} \\ (N-2)/2; & N \text{ even.} \end{cases} \tag{1}$$

Then, for m a positive integer, define the sums of the even integral powers of the cosecant and of the secant, as follows:

$$C_{2m}(N) = \sum_{k=1}^{Q} \csc^{2m}(k\pi/N); \tag{2}$$

$$S_{2m}(N) = \sum_{k=1}^{Q} \sec^{2m}(k\pi/N). \tag{3}$$

The purpose of the present article is to show that the sums (2) and (3), along with similar expressions for the cotangent and the tangent, can generally be expressed as even polynomials of degree $2m$ in N, with rational coefficients. In Section IV, specific expressions will be given for the polynomial coefficients of $C_{2m}(N)$ and $S_{2m}(N)$, valid for all integral $N \geq 3$ and $1 \leq m \leq 6$.

2. DIFFERENTIAL EQUATIONS AND SPECIAL SERIES FOR $\csc^{2m} z$ AND $\sec^{2m} z$

We begin our discussion with a differential relation that is satisfied by the squares of the cosecant and secant functions, in terms of the differential operator $D \equiv d/dz$. Namely, for $m = 1, 2, 3, \ldots$:

$$\prod_{n=1}^{m} [D^2 + 4n^2] \frac{\csc^2 z}{(2m+1)!} = \csc^{2m+2} z; \tag{4}$$

$$\prod_{n=1}^{m} [D^2 + 4n^2] \frac{\sec^2 z}{(2m+1)!} = \sec^{2m+2} z. \tag{5}$$

Proof of (4) by induction: Let $f = \csc\ z$, $g = \cot\ z$ and note that $g^2 = f^2 - 1, Df = -fg, Dg = -f^2$. Then one has that $D^2 f^2 = -4f^2 + 6f^4, i.e. [D^2 + 4] f^2/6 = f^4$, which proves (4) for $m = 1$.

Now assume that (4) is true for some $m > 1$ and consider

$$D^2 f^{2m+2} = (2m+2) f^{2m+4} + (2m+2)^2 f^{2m+2} g^2 = (2m+3)(2m+2) f^{2m+4} - (2m+2)^2 f^{2m+2}.$$

Then, a slight rearrangement gives, with the help of (4), that

$$[D^2 + 4(m+1)^2] \frac{f^{2m+4}}{(2m+3)(2m+2)} = \prod_{n=1}^{m+1} [D^2 + 4n^2] \frac{f^2}{(2m+3)!} = f^{2m+4}. \tag{6}$$

This is proposition (4) for $m + 1$ so that proposition is true for all $m \geq 1$.

To prove proposition (5), take $f = \sec\ z$, $g = \tan\ z$, note that $g^2 = f^2 - 1$, and proceed in the above manner.

Now (4) and (5) may be expressed in the following form:

$$P_m(D^2)\csc^2 z = \csc^{2m+2} z;\ P_m(D^2)\sec^2 z = \sec^{2m+2} z. \tag{7}$$

In these expressions, the P_m's are polynomials of degree m and we write that, for w an indeterminate:

$$P_m(w) = (w+2^2)(w+4^2)\dots(w+(2m)^2)/(2m+1)!$$

$$\equiv \sum_{r=0}^{m} \varphi_{r,m} w^r. \tag{8}$$

In the second line of (8), the coefficients $\{\varphi_{r,m}; 0 \le r \le m; 1 \le m\}$ are readily determined. Indeed, let $s_{r,m}$ be the sum of all the possible distinct products of the following numbers,

$$\{4\cdot 1^2, 4\cdot 2^2, 4\cdot 3^2, \dots, 4\cdot m^2\},$$

taken $(m-r)$ at a time, with the convention that $s_{m,m} = 1$. Then we have that:

$$\varphi_{r,m} = \frac{s_{r,m}}{(2m+1)!}; 0 \le r \le m; 1 \le m. \tag{9}$$

Relations to Bernoulli numbers and a determination of quantities directly related to the $\varphi_{r,m}$ coefficients are given in Dilcher [3, section 3]; the interested reader is referred to that work for further details and references.

Next, recall the Mittag-Leffler series expansions of $\csc^2 z$ and $\sec^2 z$ (e.g. see formula 4.3.92 in [1]):

$$\csc^2 z = \sum_{n=-\infty}^{+\infty} (z - n\pi)^{-2}; z \ne 0, \pm\pi, \pm 2\pi, \dots \tag{10}$$

$$\sec^2 z = \sum_{n=-\infty}^{+\infty} (z - (n-1/2)\pi)^{-2}; z \ne \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots \tag{11}$$

These series are absolutely convergent everywhere, except at the points shown, and they can be differentiated under the sum sign. In the sequel, for typographical brevity, we write $\sum_n$ to denote a series over all integers n.

Insertion of the series (10) and (11) in the differential identities (4) and (5) then gives, with the help of (7) to (9):

$$\begin{aligned} \csc^{2m+2} z &= \sum_{r=0}^{m} \varphi_{r,m} D^{2r} \left[\sum_{n=-\infty}^{+\infty} (z-n\pi)^{-2} \right] \\ &= \sum_{r=0}^{m} (2r+1)!\varphi_{r,m} \sum_n (z-n\pi)^{-(2r+2)}; m \ge 1; \end{aligned} \tag{12}$$

$$\begin{aligned}\sec^{2m+2} z &= \sum_{r=0}^{m} \varphi_{r,m} D^{2r} \left[\sum_{n=-\infty}^{+\infty} (z - (n - 1/2)\pi)^{-2} \right] \\ &= \sum_{r=0}^{m} (2r+1)! \varphi_{r,m} \sum_{n} (z - (n - 1/2)\pi)^{-(2r+2)}; m \geq 1.\end{aligned} \tag{13}$$

Infinite series for the integral powers of the cosecant and secant are known. In particular, Nörlund [14, chapter 6; in German] expresses these series in terms of higher order Euler and Bernoulli polynomials and numbers. Additional references and a more recent and accessible English account on the use of these polynomials and numbers can be found in [3, sections 3 and 4].

The series given in (12) and (13) represent special rearrangements of the series for $\csc^{2m+2} z$ and $\sec^{2m+2} z$ and it will be shown below that these rearrangements result in series that can be summed exactly and yield explicit closed form expressions for the desired sums, (2) and (3). Note that the differential identities (4) and (5) are defined for $m \geq 1$ but not for $m = 0$, which is why the series (12) and (13) are limited in the same manner. It is however possible and convenient, for our purpose, to extend these series so that they apply to the case $m = 0$ as well. This can be done simply, through series (10) and (11), by introducing an extra φ-coefficient: $\varphi_{r=0,m=0} = 1$. This will allow us to evaluate $C_2(N)$ and $S_2(N)$ also and the new convention regarding the φ-coefficients will thus be adopted throughout the remainder of this article.

Now let $1 \leq k \leq Q$ be a positive integer, with Q as given in (1), and set $z = k\pi/N$ in (12) and (13). Then take into account the extra coefficient $\varphi_{r=0,m=0} = 1$, as mentioned above, and sum over the index k to get (2) and (3), thus:

$$\begin{aligned}C_{2m}(N) &= \sum_{r=0}^{m-1} (2r+1)! \varphi_{r,m-1} \sum_{k=1}^{Q} \sum_{n} (k\pi/N - n\pi)^{-(2r+2)} \\ &\equiv \sum_{r=1}^{m} (2r-1)! \varphi_{r-1,m-1} J_{2r}(N);\ m \geq 1;\end{aligned} \tag{14}$$

$$\begin{aligned}S_{2m}(N) &= \sum_{r=0}^{m-1} (2r+1)! \varphi_{r,m-1} \sum_{k=1}^{Q} \sum_{n} (k\pi/N - (n - 1/2)\pi)^{-(2r+2)} \\ &\equiv \sum_{r=1}^{m} (2r-1)! \varphi_{r-1,m-1} K_{2r}(N);\ m \geq 1.\end{aligned} \tag{15}$$

Note that we have shifted the index r in the last lines of (14) and (15) and these relations can also be used to find $C_2(N)$ and $S_2(N)$. The following infinite series were defined, for integral $r \geq 1$:

$$J_{2r}(N) = \left(\frac{N}{\pi}\right)^{2r} \sum_{k=1}^{Q} \sum_{n} (k - nN)^{-2r}; \tag{16}$$

$$K_{2r}(N) = \left(\frac{N}{\pi}\right)^{2r} \sum_{k=1}^{Q} \sum_{n} \left(k - (2n-1)\frac{N}{2}\right)^{-2r}. \tag{17}$$

These infinite series, which will be evaluated next, are absolutely convergent for all the values of k in the summation interval, $1 \le k \le Q$, and they can thus be rearranged as needed.

3. EVALUATING THE SERIES $J_{2r}(N)$ AND $K_{2r}(N)$ FOR $r \ge 1$

In this section, we evaluate the series (16) and (17), which are valid for all integers $r \ge 1$, by rearranging them as follows:

$$J_{2r}(N) = \left(\frac{N}{\pi}\right)^{2r} \sum_{k=1}^{Q} [k^{-2r} + (N-k)^{-2r} + (N+k)^{-2r} + (2N-k)^{-2r} + (2N+k)^{-2r} + \ldots]; \tag{18}$$

$$K_{2r}(N) = \left(\frac{N}{\pi}\right)^{2r} \sum_{k=1}^{Q} \left[\left(\frac{N}{2} - k\right)^{-2r} + \left(\frac{N}{2} + k\right)^{-2r} + \left(\frac{3N}{2} - k\right)^{-2r} + \left(\frac{3N}{2} + k\right)^{-2r} + \ldots\right]. \tag{19}$$

As will be seen, the sums $C_{2m}(N)$ assume the same values as $S_{2m}(N)$ if N is even. For the case where N is odd, however, the two sums differ. Two cases are thus to be considered:

Case 1: $N = 2Q + 2$ is even; **Case 2:** $N = 2Q + 1$ is odd.

We now consider these cases, in turn.

Case 1: $N = 2Q + 2$ **is even.**

Upon summing over k, the terms within square brackets in (18) are the terms k^{-2r}, for all the positive integers, except for gaps at the positive integral multiples of $Q + 1 = N/2$, that is: $N/2, 2N/2, 3N/2$, etc. It then follows from the known properties of the Riemann Zeta function, $\zeta(z)$, evaluated at $z = 2r, 1 \le r$, that:

$$\begin{aligned} J_{2r}(N) &= \left(\frac{N}{\pi}\right)^{2r} \left[\sum_{k=1}^{\infty} k^{-2r} - \sum_{k=1}^{\infty} \left(\frac{kN}{2}\right)^{-2r}\right] \\ &= (N^{2r} - 2^{2r}) \frac{\zeta(2r)}{\pi^{2r}} \\ &= \frac{2^{2r-1}|B_{2r}|}{(2r)!}(N^{2r} - 2^{2r}); r \ge 1, N \text{ even.} \end{aligned} \tag{20}$$

In this expression, $\zeta(2r) = (2\pi)^{2r}|B_{2r}|/2(2r)!$ (see [1], formula 23.2.16) and $|B_{2r}|$ is the absolute value of a Bernoulli number (see [1]).

One can sum (19) in similar fashion. All the positive values of k in k^{-2r} are again covered, except for gaps at all the positive integral multiples of $N/2$. This is precisely what was found in the previous case and it then follows that $K_{2r}(N) = J_{2r}(N)$ for all even $N > 3$. Thus:

$$K_{2r}(N) = \frac{2^{2r-1}|B_{2r}|}{(2r)!}(N^{2r} - 2^{2r}); r \geq 1, \ N \text{ even} \tag{21}$$

Case 2: $N = 2Q + 1$ **is odd.**

For $J_{2r}(N)$, the steps are the same as for Case 1 above and the terms within the square brackets in (18) are again the terms k^{-2r}, except that the gaps are now found at all the positive integral multiples of $2Q + 1 = N$; that is at: $N, 2N, 3N$, etc .... It thus follows that

$$\begin{aligned} J_{2r}(N) &= \left(\frac{N}{\pi}\right)^{2r} \left[\sum_{k=1}^{\infty} k^{-2r} - \sum_{k=1}^{\infty} (kN)^{-2r}\right] \\ &= (N^{2r} - 1)\frac{\zeta(2r)}{\pi^{2r}} \\ &= \frac{2^{2r-1}|B_{2r}|}{(2r)!}(N^{2r} - 1); r \geq 1, N \text{ odd.} \end{aligned} \tag{22}$$

In this same case of N odd, we get that the terms within the square brackets of (19) are the terms $2^{2r}(2k-1)^{-2r}$, for all positive integral values of k, except for gaps at the odd integral multiples of $2Q + 1 = N$. As a result, since $1 + 3^{-2r} + 5^{-2r} + 7^{-2r} + \cdots = \zeta(2r)(1 - 2^{-2r})$, it follows that:

$$\begin{aligned} K_{2r}(N) &= \left(\frac{N}{\pi}\right)^{2r} 2^{2r} \left[\sum_{k=1}^{\infty} (2k-1)^{-2r} - \sum_{k=1}^{\infty} ((2k-1)N)^{-2r}\right] \\ &= \frac{(2^{2r} - 1)(N^{2r} - 1)\zeta(2r)}{\pi^{2r}} \\ &= \frac{2^{2r-1}(2^{2r} - 1)|B_{2r}|}{(2r)!}(N^{2r} - 1); r \geq 1, N \text{ odd.} \end{aligned} \tag{23}$$

We are now in a position to examine the desired sums for any integers $m \geq 1$ and $N \geq 3$, and to exhibit the general polynomial structure of the results. We will also demonstrate the applicability of the present methods by calculating the rational coefficients of the polynomials for $C_{2m}(N)$ and $S_{2m}(N)$, with $1 \leq m \leq 6$, for all $N \geq 3$.

4. POLYNOMIAL FORMS FOR $C_{2m}(N)$ AND $S_{2m}(N)$, WITH APPLICATIONS

With the help of (14), (15), and (20) to (23), we obtain general expressions for the desired sums, $C_{2m}(N)$ and $S_{2m}(N)$:

$$C_{2m}(N) = \sum_{r=1}^{m} a_{r,m}(N^{2r} - 2^{2r}); 1 \leq m; \ N \text{ even.} \tag{24}$$

$$C_{2m}(N) = \sum_{r=1}^{m} a_{r,m}(N^{2r} - 1); 1 \leq m;\ N \text{ odd.} \tag{25}$$

$$S_{2m}(N) = \sum_{r=1}^{m} a_{r,m}(N^{2r} - 2^{2r}); 1 \leq m;\ N \text{ even.} \tag{26}$$

$$S_{2m}(N) = \sum_{r=1}^{m} b_{r,m}(N^{2r} - 1); 1 \leq m;\ N \text{ odd.} \tag{27}$$

$$a_{r,m} = \frac{2^{2r-2}|B_{2r}|\varphi_{r-1,m-1}}{r}; b_{r,m} = (2^{2r} - 1)a_{r,m};\ 1 \leq r \leq m;\ 1 \leq m. \tag{28}$$

The above expressions are even polynomials of degree $2m$ in N, with rational coefficients, $\{a_{r,m} : 1 \leq r \leq m; 1 \leq m\}, \{b_{r,m} : 1 \leq r \leq m; 1 \leq m\}$. The coefficients can be determined from the Bernoulli numbers and the φ-coefficients, as defined through (8), (9), with the extra term: $\varphi_{r=0,m=0} = 1$. This constitutes the proof of our earlier claim about the rational polynomial character of the sums defined in (2) and (3), which is one of the main results of this article.

The following general observations are noteworthy:

1) $S_{2m}(N) = C_{2m}(N)$ when N is even; see (24) and (26);

2) the polynomial expansions for $C_{2m}(N)$, with N even, and those for $C_{2m}(N)$, with N odd, differ only by their constant term; see (24) and (25).

We now determine the following set of polynomial coefficients to illustrate how the method works in practice: $\{a_{r,m}, b_{r,m} : 1 \leq r \leq m; 1 \leq m \leq 6\}$.

From (28) and (9), plus the definition of the extra coefficient $\varphi_{r=0,m=0} = 1$, the $a_{r,m}$ and $b_{r,m}$ coefficients may be written as:

$$a_{r,m} = \frac{2^{2r-2}|B_{2r}|s_{r-1,m-1}}{r(2m-1)!}; b_{r,m} = (2^{2r} - 1)a_{r,m}; s_{r-1=0,m-1=0} = 1; 1 \leq r \leq m; 1 \leq m. \tag{29}$$

To determine the desired coefficients for $1 \leq m \leq 6$ and for arbitrary $N \geq 3$, we need the appropriate Bernoulli numbers, as well as the numbers $s_{r-1,m-1}$. These are given in the following two lists.

Absolute values of even-indexed Bernoulli numbers, from B_2 to B_{12}:

$$|B_2| = \frac{1}{6}; |B_4| = \frac{1}{30}; |B_6| = \frac{1}{42}; |B_8| = \frac{1}{30}; |B_{10}| = \frac{5}{66}; |B_{12}| = \frac{691}{2730}.$$

Sums of products, $s_{r-1,m-1}$, from $s_{0,0}$ to $s_{6,6}$ (columns, $r-1$; rows, $m-1; 1 \leq m \leq 6$):

$\frac{r-1}{m-1}$	0	1	2	3	4	5
0	1					
1	4	1				
2	64	20	1			
3	2,304	784	56	1		
4	147,456	52,480	4,368	120	1	
5	14,745,600	5,395,456	489,280	16,368	220	1

Finally, with the above lists of values and with (29), we now get the polynomial coefficients for $C_{2m}(N)$ and $S_{2m}(N)$, with $1 \leq m \leq 6$, for all $N \geq 3$.

Table of $a_{r,m}$ coefficients, from $a_{1,1}$ to $a_{6,6}$ (columns, r; rows, m):

$\frac{r}{m}$	1	2	3	4	5	6
1	$\frac{1}{6}$					
2	$\frac{1}{9}$	$\frac{1}{90}$				
3	$\frac{4}{45}$	$\frac{1}{90}$	$\frac{1}{945}$			
4	$\frac{8}{105}$	$\frac{7}{675}$	$\frac{4}{2{,}835}$	$\frac{1}{9{,}450}$		
5	$\frac{64}{945}$	$\frac{82}{8{,}505}$	$\frac{13}{8{,}505}$	$\frac{1}{5{,}670}$	$\frac{1}{93{,}555}$	
6	$\frac{128}{2{,}079}$	$\frac{1{,}916}{212{,}625}$	$\frac{278}{178{,}605}$	$\frac{31}{141{,}750}$	$\frac{2}{93{,}555}$	$\frac{691}{638{,}512{,}875}$

The $b_{r,m}$ coefficients are found from (29) with the help of the corresponding element in the list of $a_{r,m}$ coefficients, through $b_{r,m} = (2^{2r} - 1)a_{r,m}$:

$$b_{1,m} = 3a_{1,m}; b_{2,m} = 15a_{2,m}; b_{3,m} = 63a_{3,m}; b_{4,m} = 255a_{4,m}; b_{5,m} = 1023a_{5,m}; b_{6,m} = 4095a_{6,m}.$$

Additional specific results may readily be obtained in the same manner.

The expressions for $C_2(N)$ and for $S_2(N)$ are known, albeit in a different form, for N odd or even; they can be found as formulas 4.4.6.4 and 4.4.6.8 in [15], for example. We have been unable to locate references to formulas for $C_{2m}(N)$ and for $S_{2m}(N)$ where $m > 1$, however. It may thus well be that most of the above results are new.

4. POLYNOMIAL FORMS FOR THE EVEN POWERS OF COTANGENT AND TANGENT

As mentioned in Section 1, there are closely related formulas for the corresponding sums of even powers of the tangent and cotangent functions. These sums stem from the known trigonometric identities: $\tan^2 z = \sec^2 z - 1$ and $\cot^2 z = \csc^2 z - 1$. We give these relations here, for completeness.

Setting $z = k\pi/N$ in the identities above, then raising each expression to the m^{th} power, the binomial expansion theorem gives:

$$\begin{aligned}
\cot^{2m}\left(\frac{k\pi}{N}\right) &= \left[\csc^2\left(\frac{k\pi}{N}\right) - 1\right]^m \\
&= \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\csc^{2r}\left(\frac{k\pi}{N}\right); \\
\tan^{2m}\left(\frac{k\pi}{N}\right) &= \left[\sec^2\left(\frac{k\pi}{N}\right) - 1\right]^m \\
&= \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\sec^{2r}\left(\frac{k\pi}{N}\right).
\end{aligned}$$

Then summing each of these identities over k, with $1 \leq k \leq Q$, we obtain the following relations, using the definitions in (2) and (3):

$$\sum_{k=1}^{Q}\cot^{2m}\left(\frac{k\pi}{N}\right) = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}C_{2r}(N); \tag{30}$$

$$\sum_{k=1}^{Q}\tan^{2m}\left(\frac{k\pi}{N}\right) = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}S_{2r}(N). \tag{31}$$

Formulas (30) and (31) demonstrate that these sums are indeed polynomials of degree $2m$ in N, with rational coefficients also, as claimed in the Introduction. Specific formulas are readily handled and we skip further details.

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