IDENTITIES FOR THE GENERATING FUNCTION OF THE MULTISET $\lfloor n\Phi^m \rfloor$ FOR $m = -1, 1, 2$

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ABSTRACT

Some remarkable identities involving the power series in which the exponents are of form $\lfloor n\alpha \rfloor$ with some irrational $\alpha > 0$ have been obtained. Here we present short proofs for some related identities with $\alpha = \Phi^m, m = -1, 1, 2$.

1. INTRODUCTION

Among others, Adams and Davison [1] and Anderson, Brown, and Shiue [2] considered the power series

$$\sum_{n=1}^{\infty} z^{\lfloor n\alpha \rfloor},$$

$|z| < 1$, and derived identities in terms of continued fraction expansions with partial quotients that are rational functions of z. This power series is also referred to as the generating function for the spectrum of the multiset $\lfloor n\alpha \rfloor$.

Most prominently, they showed that for $b > 1$ integer, $S_b(\alpha) = (b-1)\sum_{n=1}^{\infty} \frac{1}{b^{\lfloor n/\alpha \rfloor}}$ can be described explicitly as the infinite simple continued fraction $[t_0, t_1, \dots]$, with $\alpha = [a_0, a_1, \dots]$, n^{th} convergents $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n], n \geq 0$, $q_{-1} = 0$, and $t_0 = a_0 b, t_n = \frac{b^{q_n} - b^{q_{n-2}}}{b^{q_{n-1}} - 1}, n \geq 1$. Clearly,

$$\frac{z}{1-z} S_{1/z}(1/\alpha) = \sum_{n=1}^{\infty} z^{\lfloor n\alpha \rfloor} \tag{1}$$

for any z which is the reciprocal of an integer $b > 1$.

In [3], the remarkable identity

$$\frac{z^{F_1}}{1 + \frac{Z^{F_2}}{1 + \frac{z^{F_3}}{1 + \cdots}}} = (1-z)\sum_{n=1}^{\infty} z^{\lfloor n\Phi \rfloor},$$

$|z| < 1$, or in an equivalent form (using a slightly unusual continued fraction form corresponding to the so called continuant polynomials)

$$[0, z^{-F_0}, z^{-F_1}, z^{-F_2}, \dots] = \frac{1-z}{z} \sum_{n=1}^{\infty} z^{\lfloor n\Phi \rfloor}, \tag{2}$$

was proven. According to [2], it can be used to derive the power series version of the identity (1) for $\alpha = \Phi$.

2. RESULTS

We will consider and prove two specific power series identities for $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=1}^{\infty} z^{\lfloor \frac{n}{\Phi} \rfloor} - \sum_{n=1}^{\infty} z^{\lfloor n\Phi \rfloor} \tag{3}$$

and

$$\frac{1+z}{1-z} = \sum_{n=1}^{\infty} z^{\lfloor \frac{n}{\Phi} \rfloor} + \sum_{n=1}^{\infty} z^{\lfloor n\Phi^2 \rfloor}. \tag{4}$$

To obtain a partial proof of (3) for any $z = 1/b, b > 1$ integer, we can use identity (1)

$$\frac{z}{1-z}\big(S_{1/z}(\Phi) - S_{1/z}(1/\Phi)\big) = \frac{z}{1-z}\left(\left[\frac{1}{z}, A\right] - [0, A]\right) = \frac{1}{1-z}$$

with A comprising the "partial quotients" $\frac{(\frac{1}{z})^{F_{n+1}} - (\frac{1}{z})^{F_{n-1}}}{(\frac{1}{z})^{F_n} - 1}, n \geq 1$.

To get a direct proof without using identities (1) and (2), we observe that in the first sum, each term z^k comes with a coefficient 1 or 2, and $k = \lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor$ if and only if $\lfloor (n - \lfloor \frac{n}{\Phi} \rfloor)\Phi \rfloor = k$. In fact, if $\lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor$ then

$$\frac{n}{\Phi} \leq \left(n - \left\lfloor \frac{n}{\Phi} \right\rfloor\right)\Phi = \left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right)\Phi < n\Phi + \Phi - (n+1) = (n+1)(\Phi - 1) = \frac{n+1}{\Phi}$$

since $\frac{1}{\Phi} = \Phi - 1$, $0 \leq \lfloor \frac{a}{b} \rfloor b \leq a$, and $0 \leq a - \lfloor \frac{a}{b} \rfloor b < b$ with $a, b \geq 0$, and we can take the integer parts. In other words, there is a term z^k to be subtracted in the second sum in (3). On the other hand, if $k = \lfloor r\Phi \rfloor$ for some integer $r \geq 1$ (i.e., $k-1$ is *Fibonacci even* [3]) then with $n = \lfloor k\Phi \rfloor + 1$ we get $\lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor = k$ to the same effect.

To prove identity (4), we observe that, as $\frac{1}{\Phi} + \frac{1}{\Phi^2} = 1$, Beatty's theorem guarantees that $a_n = \lfloor n\Phi \rfloor$ and $b_n = \lfloor n\Phi^2 \rfloor = a_n + n,\ n \geq 1$, form a set in which each positive integer occurs precisely once. This yields

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} z^{\lfloor n\Phi \rfloor} + \sum_{n=1}^{\infty} z^{\lfloor n\Phi^2 \rfloor}, \tag{5}$$

which implies (4) via (3).

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