THE INVERSE OF A FINITE SERIES AND A THIRD-ORDER RECURRENT SEQUENCE

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ABSTRACT

It is apparently not well-known that $g(n) = f(n) + f(n-1) + f(n-2)$ if and only if

$$f(n) = \sum_{k=0}^{n} g(n-k) \sum_{j=0}^{[\frac{k}{2}]} (-1)^{k-j} \binom{k-j}{j},$$

where we suppose that $f(n) = 0$ for $n < 0$. This may also be expressed as

$$f(n) = \sum_{k=0}^{n} (-1)^k g(n-k) \frac{1}{2} \left((-1)^{[\frac{k}{3}]} + (-1)^{[\frac{k+1}{3}]} \right).$$

We show how to solve for $f(n)$ in the general case

$$g(n) = \sum_{k=0}^{r} f(n-k), \text{ where } f(n) = 0 \text{ for } n < 0, \text{ with } 1 \le r \le n.$$

We shall also see that the values at which g is evaluated in forming the inverse satisfy a third-order recurrence relation of the form

$$a_n = a_{n-1} + a_{n-2} - a_{n-3}.$$

1. INTRODUCTION

It is easy to verify that

$$g(n) = f(n) + f(n-1), \tag{1}$$

if and only if

$$f(n) = \sum_{k=0}^{n} (-1)^{n-k} g(k), \tag{2}$$

where it is assumed that $f(n) = 0$ for $n < 0$. It is perhaps less obvious and not well-known that

$$g(n) = f(n) + f(n-1) + f(n-2) \tag{3}$$

if and only if

$$f(n) = \sum_{k=0}^{n} g(n-k) \sum_{j=0}^{[\frac{k}{2}]} (-1)^{k-j} \binom{k-j}{j} \tag{4}$$

where again we suppose that $f(n) = 0$ for $n < 0$. By means of a formula due to Schwatt [6, p. 4] this may be expressed using the bracket function in the alternative form

$$f(n) = \sum_{k=0}^{n} (-1)^k g(n-k) \frac{1}{2} \left((-1)^{[\frac{k}{3}]} + (-1)^{[\frac{k+1}{3}]} \right). \tag{5}$$

Furthermore, formula (5) may also be expressed in the quite different looking form

$$f(n) = \sum_{k=0}^{n} A_k \left(\left[\frac{k}{3} \right] - \left[\frac{k-2}{3} \right] \right) g(n-k), \tag{6}$$

where A_k equals -1 if 3 divides $k-1$ and equals 1 otherwise.

What happens is that not every value of $g(j),\ 0 \le j \le n$, is used when forming the inverse of (3). I am unable to provide any reference in the literature for these three formulas as the inverse of the series (3). All three will be proved below.

These inverse pairs suggest the general problem of how to solve for $f(n)$ when

$$g(n) = \sum_{k=0}^{r} f(n-k),\ \text{where } f(n) = 0 \text{ for } n < 0,\ \text{with } 1 \le r \le n. \tag{7}$$

It is not difficult to see that $f(n)$ will turn out to be a linear combination of the g's, but the law of formation of the coefficients is perhaps not so obvious. Interestingly enough, we shall also see that the values at which g is evaluated in forming the inverse satisfy a third-order recurrence relation of the form $a_n = a_{n-1} + a_{n-2} - a_{n-3}$. The object of our paper is to give a complete solution to the problem of inverting (7).

2. A LOOK AT SOME SPECIAL CASES

It is interesting to examine some special cases of the inversion of (7).

$$\begin{aligned}
&f(0) = g(0),\\
&f(1) = g(1) - g(0),\\
&f(2) = g(2) - g(1) + g(0),\\
&f(3) = g(3) - g(2) + g(1) - g(0),\\
&f(4) = g(4) - g(3) + g(2) - g(1) + g(0),\\
&f(5) = g(5) - g(4) + g(3) - g(2) + g(1) - g(0),\\
&f(6) = g(6) - g(5) + g(4) - g(3) + g(2) - g(1) + g(0),
\end{aligned}$$

Table 1. Solutions of $g(n) = f(n) + f(n-1)$.

When $r = 1$ we find the array shown in Table 1. The pattern continues in accord with formula (2), whose proof is easy by mathematical induction. When $r = 2$, we find the array shown in Table 2.

$$\begin{aligned}
f(0) &= g(0),\\
f(1) &= g(1) - g(0),\\
f(2) &= g(2) - g(1),\\
f(3) &= g(3) - g(2) + g(0),\\
f(4) &= g(4) - g(3) + g(1) - g(0),\\
f(5) &= g(5) - g(4) + g(2) - g(1),\\
f(6) &= g(6) - g(5) + g(3) - g(2) + g(0),\\
f(7) &= g(7) - g(6) + g(4) - g(3) + g(1) - g(0),\\
f(8) &= g(8) - g(7) + g(5) - g(4) + g(2) - g(1),\\
f(9) &= g(9) - g(8) + g(6) - g(5) + g(3) - g(2) + g(0),\\
f(10) &= g(10) - g(9) + g(7) - g(6) + g(4) - g(3) + g(1) - g(0),
\end{aligned}$$

Table 2. Solutions of $g(n) = f(n) + f(n-1) + f(n-2)$.

Here, we find that g is evaluated only at certain numbers between 0 and n. We always have n and $n-1$ and then every third number down is absent.

Consider the case $n = 9$. Starting with 9, 8, and 6 the next value to be used is found from $6+8-9=5$. The next value to be used will be $5+6-8=3$. The next value will be $3+5-6=2$, and finally $2+3-5=0$. The process terminates when we reach 0 or sometimes 1, no negatives being allowed. The recursive pattern of the form $a_i = a_{i-1} + a_{i-2} - a_{i-3}$ continues. The plus and minus signs alternate nicely.

$$\begin{aligned}
f(0) &= g(0),\\
f(1) &= g(1) - g(0),\\
f(2) &= g(2) - g(1),\\
f(3) &= g(3) - g(2),\\
f(4) &= g(4) - g(3) + g(0),\\
f(5) &= g(5) - g(4) + g(1) - g(0),\\
f(6) &= g(6) - g(5) + g(2) - g(1),\\
f(7) &= g(7) - g(6) + g(3) - g(2),\\
f(8) &= g(8) - g(7) + g(4) - g(3) + g(0),\\
f(9) &= g(9) - g(8) + g(5) - g(4) + g(1) - g(0),\\
f(10) &= g(10) - g(9) + g(6) - g(5) + g(2) - g(1),\\
f(11) &= g(11) - g(10) + g(7) - g(6) + g(3) - g(2),\\
f(12) &= g(12) - g(11) + g(8) - g(7) + g(4) - g(3) + g(0),\\
f(13) &= g(13) - g(12) + g(9) - g(8) + g(5) - g(4) + g(1) - g(0),
\end{aligned}$$

Table 3. Solutions of $g(n) = f(n) + f(n-1) + f(n-2) + f(n-3)$.

When $r = 3$, we find the array shown in Table 3.

Again, for example when $n = 13$, the successive values where g is to be evaluated are found by the same recursive formula. Thus, after 9, we next have $9 + 12 - 13 = 8$, then $8+9-12 = 5$, then $5+8-9 = 4, 4+5-8 = 1$, and finally $1+4-5 = 0$. Again the plus and minus signs are regular.

When $r = 4$, let us agree to just write the numbers where g is to be evaluated, but omit the plus or minus sign to be used, and we may abbreviate the array of g coefficients as shown in Table 4.

0								
1	1							
2	1							
3	2							
4	3							
5	4	0						
6	5	1	0					
7	6	2	1					
8	7	3	2					
9	8	4	3					
10	9	5	4	0				
11	10	6	5	1	0			
12	11	7	6	2	1			
13	12	8	7	3	2			
14	13	9	8	4	3			
15	14	10	9	5	4	0		
16	15	11	10	6	5	1	0	
17	16	12	11	7	6	2	1	
18	17	13	12	8	7	3	2	
19	18	14	13	9	8	4	3	
20	19	15	14	10	9	5	4	0

Table 4. Coefficients for solutions of $g(n) = f(n) + f(n-1) + f(n-2) + f(n-3) + f(n-4)$.

Again the numbers on a given row follow from the three initial values by using by the recurrence $a_i = a_{i-1} + a_{i-2} - a_{i-3}$. Thus we have here $15 + 19 - 20 = 14$, then $14 + 15 - 19 = 10$, $10 + 14 - 15 = 9$, etc. Where the row ends follows a curious staircase pattern.

3. THE SOLUTION OF THE $r = 2$ PROBLEM

To solve $g(n) = f(n) + f(n-1) + f(n-2)$ we use generating functions.

$$\begin{aligned}\sum_{n=0}^{\infty} g(n)t^n &= \sum_{n=0}^{\infty} f(n)t^n + \sum_{n=0}^{\infty} f(n-1)t^n + \sum_{n=0}^{\infty} f(n-2)t^n \\ &= (1+t+t^2)\sum_{n=0}^{\infty} f(n)t^n, \text{ since } f(n) = 0 \text{ for } n < 0.\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} f(n)t^n = \frac{1}{1+t+t^2} \sum_{n=0}^{\infty} g(n)t^n. \tag{8}$$

In my paper [2] the expansion

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m,x,y,p,C) \tag{9}$$

is studied, where $m \geq 1$ is an integer and the other variables are, in general, unrestricted. In particular p may be any real number. It is shown in [2] that

$$P_n(m,x,y,p,C) = \sum_{k=0}^{[\frac{n}{m}]} \binom{p}{k} \binom{p-k}{n-mk} C^{p-n+(m-1)k} y^k (-mx)^{n-mk}. \tag{10}$$

A generalized chain rule explained in [4] was used in finding (10). This formula includes many special functions such as the polynomials of Legendre, Tchebycheff, Gegenbauer, Pincherle, and Humbert and others. The case we need is when $C = 1, m = 2, x = -1/2, y = 1$, and $p = -1$. This actually gives us the Gegenbauer polynomial $U_n(-1/2)$, and in fact

$$\begin{aligned} P_n(2,-1/2,1,-1,1) &= \sum_{k=0}^{[\frac{n}{2}]} \binom{-1}{k} \binom{-1-k}{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{-1-k}{n-2k} \\ &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^{n-k} \binom{n-k}{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^{n-k} \binom{n-k}{k} \end{aligned}$$

where we have used the formulas

$$\binom{-1}{k} = (-1)^k \quad \text{and more generally} \quad \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}.$$

Applying our formula for P_n to relation (8) we get

$$\begin{aligned} \sum_{n=0}^{\infty} f(n)t^n &= \sum_{n=0}^{\infty} g(n)t^n \sum_{k=0}^{\infty} t^k \sum_{j=0}^{[\frac{k}{2}]} (-1)^{k-j} \binom{k-j}{k} \\ &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} g(n-k) \sum_{j=0}^{[\frac{k}{2}]} (-1)^{k-j} \binom{k-j}{j}, \end{aligned}$$

and upon equating coefficients of t^n we have proved formula (4).

Schwatt arrives at the expansion of $(1+t+t^2)^{-1}$ in a different manner and then observes that we may also write

$$\frac{1}{1+t+t^2} = \frac{1-t}{1-t^3} = \sum_{a=0}^{\infty}(-1)^a \sum_{m=0}^{\infty} t^{3m+a}, \tag{11}$$

and he is led to the binomial identity

$$\sum_{j=0}^{[\frac{k}{2}]}(-1)^j \binom{k-j}{j} = \frac{1}{2}\left((-1)^{[\frac{k}{3}]} + (-1)^{[\frac{k+1}{3}]}\right). \tag{12}$$

This identity is formula (1.75) in my book [3]. Upon equating coefficients we are led to the formula (5). However, it is also easy to show that $\left[\frac{k}{3}\right] - \left[\frac{k-2}{3}\right]$ does the same trick of skipping numbers in the same way, but we must multiply it by the factor A_k used in relation (6) to get the proper signs. More about this when we examine the general case.

4. THE GENERAL CASE

We use generating functions to analyze

$$g_r(n) = \sum_{k=0}^{r} f(n-k), \qquad \text{where} \qquad r = 0, 1, 2, 3, \ldots .$$

In the same way that we found (8) we now get

$$\begin{aligned}\sum_{n=0}^{\infty} g_r(n)t^n &= \sum_{n=0}^{\infty} f(n)t^n + \sum_{n=0}^{\infty} f(n-1)t^n + \cdots + \sum_{n=0}^{\infty} f(n-r)t^n \\ &= (1+t+t^2+\cdots+t^r)\sum_{n=0}^{\infty} f(n)t^n,\end{aligned}$$

whence

$$\sum_{n=0}^{\infty} f(n)t^n = (1+t+t^2+\cdots+t^r)^{-1}\sum_{n=0}^{\infty} g_r(n)t^n, \tag{13}$$

so that to find the coefficients we need we must have a suitable expansion for the reciprocal of the finite geometric series. The formulas in [2] no longer apply since they hold only for $r = 2$.

One is tempted to use the formula of Schwatt [6, p. 119-120], which says that

$$\begin{aligned}(1+t+t^2+\cdots+t^{r-1})^p &= \sum_{n=0}^{\infty} C_r(p,n)t^n \\ &= 1 + \sum_{n=1}^{\infty} t^n \sum_{k=1}^{[\frac{n}{r}]}(-1)^k \binom{p}{k}\binom{n-rk+p-1}{n-rk}.\end{aligned} \tag{14}$$

In case p is non-negative the infinite series stops at $n = p(r-1)$.

From this we could say that

$$(1+t+t^2+\cdots+t^{r-1})^{-1} = \sum_{n=0}^{\infty} C_r(-1,n)t^n \tag{15}$$

and have

$$\begin{aligned}\sum_{n=0}^{\infty} f(n)t^n &= \sum_{n=0}^{\infty} C_r(-1,n)t^n \sum_{k=0}^{\infty} g_r(k)t^k \\ &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} C_r(-1,k)g_r(n-k),\end{aligned}$$

so that we would have

$$f(n) = \sum_{k=0}^{n} C_r(-1,k)g_r(n-k), \tag{16}$$

thereby giving $f(n)$ as a linear combination of the g's using a binomial coefficient summation for $C_r(-1,k)$.

Schwatt claimed that (14) holds for any real value of p, but in fact the formula seems to fail for $p=-1$ which we need. Incidentally, a variant proof of (14) for non-negative integer p is given in my paper [5].

It turns out that we can get the desired $C_r(p,k)$ expansion by the following method when $p=-1$. We find

$$\begin{aligned}(1+t+t^2+\cdots+t^r)^{-1} &= \frac{1}{1+(t+t^2+t^3+\cdots+t^r)} \\ &= \sum_{n=0}^{\infty}(-1)^n(t+t^2+t^3+\cdots+t^r)^n \\ &= \sum_{n=0}^{\infty}(-1)^n t^n(1+t+t^2+\cdots+t^{r-1})^n\end{aligned}$$

and we may now apply the expansion of Schwatt to the n^{th} power of the sum of the t's. We find

$$\begin{aligned}(1+t+t^2+\cdots+t^r)^{-1} &= \sum_{n=0}^{\infty}(-1)^n t^n \sum_{k=0}^{(r-1)n} C_r(n,k)t^k \\ &= \sum_{k=0}^{\infty} \sum_{\frac{kr}{r-1}\le n\le\infty} (-1)^{n-k}C_r(n-k,k)t^n \\ &= \sum_{n=0}^{\infty} C_r(-1,n)t^n.\end{aligned}$$

Thus the coefficient of t^n in this is found to be

$$\begin{aligned} C_r(-1,n) &= \sum_{k=0}^{\infty}(-1)^{n-k}C_r(n-k,k) \\ &= \sum_{k=0}^{\infty}(-1)^{n-k}\sum_{j=0}^{[\frac{k}{r}]}(-1)^j\binom{n-k}{j}\binom{n-1-rj}{n-j-k} \end{aligned}$$

but which is not very useful.

The desire to get a binomial coefficient summation in the general case is illusory, however, because formula (6) generalizes completely. We have in fact the following general result:

Theorem 1: Let $1 \le r \le n$ and $n \ge 0$ be arbitrary integers. Suppose $f(n) = 0$ whenever $n < 0$. Then the following series pairs are equivalent:

$$g_r(n) = \sum_{k=0}^{r} f(n-k). \tag{17}$$

if and only if

$$f(n) = \sum_{k=0}^{n} A_k(r)g_r(n-k)D_k(r) \tag{18}$$

where

$$D_k(r) = \left[\frac{k}{r+1}\right] - \left[\frac{k-2}{r+1}\right], \tag{19}$$

and

$$A_k(r) = \begin{cases} -1 & \text{when } r+1 \text{ divides } k-1 \\ 1 & \textit{otherwise.} \end{cases} \tag{20}$$

Moreover it is interesting to note that when r is odd we may replace (20) by the simpler formula $A_k = (-1)^k$.

Proof: We recall that

$$\sum_{n=0}^{\infty} f(n)t^n = (1+t+t^2+\cdots+t^r)^{-1}\sum_{n=0}^{\infty} g_r(n)t^n.$$

But this also says that

$$\begin{aligned}\sum_{n=0}^{\infty} f(n)t^n &= \frac{1-t}{1-t^{r+1}} \sum_{n=0}^{\infty} g_r(n)t^n \\ &= (1-t)(1-t^{r+1})^{-1} \sum_{n=0}^{\infty} g_r(n)t^n \\ &= (1-t) \sum_{k=0}^{\infty} t^{(r+1)k} \sum_{n=0}^{\infty} g_r(n)t^n \\ &= \sum_{k=0}^{\infty} t^{(r+1)k} \sum_{n=0}^{\infty} g_r(n)t^n - \sum_{k=0}^{\infty} t^{(r+1)k} \sum_{n=0}^{\infty} g_r(n)t^{n+1}.\end{aligned} \tag{21}$$

When we multiply and equate coefficients, as in formula (18), we find that we need a way to express the two facts that: (i.) we always get two successive 1's followed by $r-1$ zeroes, and that this pattern repeats; and (ii.) the terms alternate in sign. Using $D_k(r) = \left[\frac{k}{r+1}\right] - \left[\frac{k-2}{r+1}\right]$ we can get (i.). To get (ii.) we define A_k as follows: $A_k = -1$ if $r+1$ divides $k-1$ and $A_k = 1$ otherwise. When r is odd it is not difficult to see that the simpler formula for A_k works.

This completes the proof of Theorem 1.

If we choose $r = n$ we find that only the term corresponding to $k = 0$ has non-zero coefficients, and we recover the very familiar inverse series pair

$$g(n) = \sum_{k=0}^{n} f(k) \tag{22}$$

if and only if $f(n)$ is the first difference of $g(n)$, i.e.

$$f(n) = g(n) - g(n-1) = \Delta g(n). \tag{23}$$

Relation (22) is sometimes considered to be a discrete integral of the first difference of g.

5. THE THIRD ORDER RECURRENCE

In the first two sections of this paper, and in Theorem 1, we have seen that in the inverse expansion (18) g is evaluated only at certain values of its argument. This skipping is carried out according to a third order recurrence, and we now establish the following general result.

Theorem 2: Let $a_0 = a, a_1 = b$, and $a_2 = c$. Then the third order recurrence relation

$$a_{n+3} = a_{n+2} + a_{n+1} - a_n \tag{24}$$

has the explicit solution

$$a_{2n} = nc + a - na = n(c-a) + a, \tag{25}$$

$$a_{2n+1} = nc + b - na = n(c-a) + b. \tag{26}$$

This is easily proved by mathematical induction.

Corollary:

$$a_{2n+1} - a_{2n} = b - a, \tag{27}$$

$$a_{2n} - a_{2n-1} = c - b. \tag{28}$$

This result allows us to figure the skipping of terms in relation (18) by the simple algorithm of starting with $a = n, b = n - 1$, and $c = n - 1 - r$ in order and using (24) to generate the remaining terms. For example with $r = 3, n = 13, n - 1 = 12, n - 4 = 9$, then the succeeding terms are 8, 5, 4, 1 and 0. We stop the process when we reach the end of the row. This example shows how to generate the last row in Table 3, which is $f(13) = g(13) - g(12) + g(9) - g(8) + g(5) - g(4) + g(1) - g(0)$. The signs have to be alternated by the device of formula (20).

Formulas (27) and (28) show how we may make whatever skips we desire. Because of this we can see that the natural numbers in order, as used in Table 1, also arise. Indeed, we merely have to let $a = 0, b = 1, c = 2$. So the natural numbers themselves satisfy the third order recurrence relation (24) trivially.

We can generate the standard sequence of odd numbers by setting $a = 1, b = 3$, and $c = 5$. The sequence of even numbers occurs when we choose $a = 0, b = 2$, and $c = 4$. If we choose $c = a$, then the sequence generated is $a, c, a, c, a, c, a, \ldots$.

It is also clear that we may have the sequence count down from a starting value or count up from a small value. Here is an example to show how we count upwards. Let us set $a = 0, b = 1$, and $c = 4$. Then the recurrence generates the infinite sequence 0, 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29, 32, 33, ... producing two consecutive numbers and then skipping two consecutive numbers.

Another example of the curious sequences we can generate is this example: 2, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 8, 7, 9, 8, 10, 9, ... Here we have two copies of the natural number sequence interwoven.

Since $a_{n+3} - a_{n+2} = a_{n+2} - a_n$, we can sum both sides and obtain the novel fact that

$$a_{n+2} = a_n + c - a. \tag{29}$$

FInally, by using the recurrence (24) it is easy to establish the generating function

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{a + (b-a)x + (c-b-a)x^2}{1 - x - x^2 + x^3} \tag{30}$$

Table 5 lists a few values of the general third order sequence so generated.

n	a_n
0	a
1	b
2	c
3	$c+b-a$
4	$2c-a$
5	$2c+b-2a$
6	$3c-2a$
7	$3c+b-3a$
8	$4c-3a$
9	$4c+b-4a$
10	$5c-4a$
11	$5c+b-5a$
12	$6c-5a$
13	$6c+b-6a$
14	$7c-6a$
15	$7c+b-7a$
16	$8c-7a$
17	$8c+b-8a$

Table 5. Short table of the general third order recurrent sequence.

6. ANOTHER FORM OF THE GENERAL SOLUTION

. The coefficients in (18) may be written in different ways. We offer

Theorem 3: The inverse of (17) may be written in the following way:

$$f(n) = \sum_{k=0}^{n} (-1)^k E_k(r) g(n-k) \quad \text{when} \quad r \text{ is even;} \tag{31}$$

$$f(n) = \sum_{k=0}^{n} (-1)^k |E_k(r)| g(n-k) \quad \text{when} \quad r \text{ is odd;} \tag{32}$$

where

$$E_k(r) = \frac{1}{2}\left\{(-1)^{\left[\frac{k}{r+1}\right]} + (-1)^{\left[\frac{k+r-1}{r+1}\right]}\right\}. \tag{33}$$

The proof is tedious algebra to show that the same coefficients are generated.

Theorem 3 using formula (33) is the generalization of (5) which was inspired by the formula of Schwatt. It is interesting, as shown by Theorems 1 and 3, that we can express the coefficients using either differences of bracket functions or as sums of bracket function powers of -1. But since the coefficients take on only the values $-1, 0$ and 1, this is not too surprising. The author has noted this to happen in the case of other enumeration formulas.

7. SOME GENERAL REMARKS ABOUT THE INVERSE FORMULA

Let us write the series and its inverse in the following way:

$$g_r(n) = \sum_{k=0}^{r} f(n-k) \tag{17}$$

if and only if

$$f(n) = \sum_{k=0}^{n} K_r(k,n) g_r(n-k) \tag{34}$$

where, as we have seen, $K_r(0,n) = 1, K_r(1,n) = -1$, and $K_r(2,n) = 0$.

It may be of interest to show an inductive proof that the coefficients exist. This can be done as follows:

$$\begin{aligned} g_r(n+1) &= \sum_{k=0}^{r} f(n+1-k) = f(n+1) + \sum_{k=1}^{r} f(n+1-k) \\ &= f(n+1) + \sum_{k=0}^{r-1} f(n-k), \end{aligned}$$

so that

$$\begin{aligned} f(n+1) = g(n+1) - \sum_{k=0}^{r-1} f(n-k) &= g(n+1) - \sum_{k=0}^{r-1}\sum_{j=0}^{n-k} K_r(j, n-k) g_r(n-k-j) \\ &= g(n+1) - \sum_{k=0}^{r-1}\sum_{j=0}^{n-k} K_r(n-k-j, n-k) g_r(j) \\ &= g(n+1) - \sum_{j=0}^{n} g(j) \sum_{k=0}^{r-1} K_r(n-k-j, n-k) \\ &= g(n+1) - \sum_{j=0}^{n} g(n-j) \sum_{k=0}^{r-1} K_r(j-k, n-k) \end{aligned}$$

which we may say is of the form

$$\sum_{j=0}^{n+1} K_r(j,n) g_r(n-j), \tag{35}$$

and from which a complicated recurrence could be developed for $K_r(k,n)$.

We have seen that $K_r(k,n) = 0$ frequently, and the next theorem tells us how many non-zero coefficients there will be in the expansion (34).

Theorem 4: The number of non-zero K coefficients in the inverse expansion (34) is given by the formula

$$N_r(n) = \left[\frac{n}{r+1}\right] + \left[\frac{n-1}{r+1}\right] + 2. \tag{36}$$

The proof is omitted.

n	$N_1(n)$	$N_2(n)$	$N_3(n)$	$N_4(n)$	$N_5(n)$	$N_6(n)$	$N_7(n)$	$N_8(n)$
0	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
2	3	2	2	2	2	2	2	2
3	4	3	2	2	2	2	2	2
4	5	4	3	2	2	2	2	2
5	6	4	4	3	2	2	2	2
6	7	5	4	4	3	2	2	2
7	8	6	4	4	4	3	2	2
8	9	6	5	4	4	4	3	2
9	10	7	6	4	4	4	4	3
10	11	8	6	5	4	4	4	4
11	12	8	6	6	4	4	4	4
12	13	9	7	6	5	4	4	4
13	14	10	8	6	6	4	4	4
14	15	10	8	6	6	5	4	4
15	16	11	8	7	6	6	4	4
16	17	12	9	8	6	6	5	4
17	18	12	10	8	6	6	6	4
18	19	13	10	8	7	6	6	5
19	20	14	10	8	8	6	6	6
20	21	14	11	9	8	6	6	6
21	22	15	12	10	8	7	6	6
22	23	16	12	10	8	8	6	6
23	24	16	12	10	8	8	6	6
24	25	17	13	10	9	8	7	6
25	26	18	14	11	10	8	8	6

Table 6. This table gives values of $N_r(n)$ for $0 \le n \le 25$ and $1 \le r \le 8$.

Table 6 presents values of $N_r(n)$ for $0 \le n \le 25$ and $1 \le r \le 8$.

Another way of writing out the inverse expansion is shown in our next result:

Theorem 5: The inverse of

$$g_r(n) = \sum_{k=0}^{r} f(n-k). \tag{17}$$

may be written in the form

$$f(n) = \sum_{k=0}^{s} (-1)^k g_r(b_k(n,r)) \tag{37}$$

where $b_k(n,r)$ satisfies the third order recurrence with $b_k(n,r) = b_{k-1}(n,r) + b_{k-2}(n,r) - b_{k-3}(n,r)$ and where $b_0(n,r) = n, b_1(n,r) = n-1$, and $b_2(n,r) = 0$, and other values may be determined readily.

ADDENDUM

We have remarked that (14) of Schwatt fails when $p = -1$. To verify that the formula may not be trusted in this case, consider the case when $r = 2$. The formula would claim that

$$\begin{aligned}(1+t)^{-1} &= 1 + \sum_{n=1}^{\infty} t^n \sum_{k=1}^{[\frac{n}{2}]} (-1)^k \binom{-1}{k} \binom{n-2k-2}{n-2k} \\ &= 1 + \sum_{n=1}^{\infty} t^n \sum_{k=1}^{[\frac{n}{2}]} (-1)^k (-1)^k \binom{n-2k-2}{n-2k} \\ &= 1 + \sum_{n=1}^{\infty} t^n \sum_{k=1}^{[\frac{n}{2}]} \binom{n-2k-2}{n-2k} \\ &= 1 + t \sum_{k=1}^{0} \binom{-2k-2}{-2k} + \sum_{n=2}^{\infty} t^n \sum_{k=1}^{[\frac{n}{2}]} \binom{n-2k-2}{n-2k},\end{aligned}$$

but the coefficient of t here is meaningless, since there is no consistent meaning that may be given to $\binom{-a}{-b}$ when a and b are positive integers. See, for example, my paper [1] where I discuss the difficulty.

Then, since we know that $(1+t)^{-1} = 1 - t + t^2 - t^3 + \dots$, we see that (14) fails for even the second term since the coefficient of t must be equal to -1. Thus it is preferable to develop the expansion in another way as we have done in our text.

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