ISODECIMAL NUMBERS

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ABSTRACT

The aim of this paper is to investigate pairs of real numbers of the type $(x, \frac{1}{x})$, $(x, \frac{a}{x})$ and (x, x^2), where the first component is a real number $x \neq 0$ and the fractional parts of the coordinates are equal. We call such numbers *isodecimal*.

1. INTRODUCTION

In section 2, we introduce the concept of an isodecimal equivalence relation on $\mathbb{R}$, the set of real numbers. Using this equivalence relation, we will give a new characterization of the golden section $\alpha = \frac{1+\sqrt{5}}{2}$ in section 5. In section 3, we determine when numbers are isodecimal to their reciprocals. In sections 4 and 6, we examine those real numbers isodecimal to a proportional number of the reciprocal or to the square of the number. Moreover, we solve the question of finding real numbers $x \neq 0$ isodecimal to $\frac{1}{x}$, corresponding to a fixed integer m in a sense that the integer part of x is equal to m. In section 6, we introduce the notion of an isodecimal point in $\mathbb{R}^2$, and we investigate when isodecimal points are points lying on the equilateral hyperbola $y = \frac{a}{x}$. In this section, we also construct a one-to-one correspondence of the equilateral hyperbola $y = \frac{a}{x}$ onto the hyperbola $y = \frac{4^n a}{x}$, for each integer $n > 0$, that preserves isodecimal points. Finally, in section 7, we investigate those isodecimal points on parabola $y = ax^2$. Moreover, using the same technique, we construct a one-to-one correspondence, σ, between the parabola $y = ax^2$ and the parabola $y = -4ax^2$. Unlike what happens for hyperbolas, such a mapping doesn't preserve the isodecimal property. Theorem 7.2 provides a necessary and sufficient condition under which the isodecimal property is preserved by both σ and σ^{-1}.

2. NOTATIONS AND USEFUL FACTS

Let x be a real number. Throughout this paper, we make use of the standard notations $\lfloor x \rfloor$ and $\lceil x \rceil$ for the floor and ceiling functions. We also let $I(x)$ be the integer part of $x, sgn(x)$ be the sign of $x, F(x)$ be the decimal/fractional part of x. Note that $I(x) = sgn(x)\lfloor |x| \rfloor$ and

$$F(x) = x - I(x) = \begin{cases} 0.a_1a_2a_3..., & \text{if } x = n.a_1a_2a_3... > 0, \\ 0, & \text{if } x = 0, \\ -0.a_1a_2a_3..., & \text{if } x = n.a_1a_2a_3... < 0. \end{cases}$$

Definition 2.1: *The real number x is said to be isodecimal to y if $F(x) = F(y)$.*

Lemma 2.1: *If x, y are real numbers and k is an integer, $k \neq 0$, then:*

i) If $sgn(x) = sgn(y)$, then x is isodecimal to y if and only if $x - y \in \mathbb{Z}$;
ii) If $sgn(x) \neq sgn(y)$, then x is isodecimal to y if and only if $x, y \in \mathbb{Z}$;
iii) If $F(x) = F(y)$ then $F(kx) = F(ky)$.

Proof: (i) If $F(x) = F(y)$ then $x - y = I(x) - I(y) \in \mathbb{Z}$. On the other hand, if $sgn(x) = sgn(y)$ and $x - y = k$, with $k \in \mathbb{Z}$, then $x = y + k$, and $I(x) = I(y) + k$. Therefore, $x - I(x) = y + k - (I(y) + k) = y - I(y)$, namely, $F(x) = F(y)$.
(ii) If $x < 0 < y$ then $F(x) \leq 0 \leq F(y)$ and the result holds if and only if $F(x) = F(y) = 0$.
(iii) This is a trivial consequence of (i) and (ii). □

3. NUMBERS ISODECIMAL TO RECIPROCAL

Let $E = \{x \in \mathbb{R} \setminus \{0\} \ : \ F(x) = F(\frac{1}{x})\}$.
Theorem 3.1: *We shall show that $x \in E$ if and only if there exists $k \in \mathbb{Z}$ such that*

$$x = \frac{k \pm \sqrt{k^2 + 4}}{2}.$$

Proof: By (i) of Lemma 2.1, $x \in E$ if and only if there exists a $k \in \mathbb{Z}$, such that x is a solution of the equation $x^2 - kx - 1 = 0$. □
Theorem 3.2: *Let $m \in \mathbb{Z}$ be fixed and let $x \in \mathbb{R} \setminus \{0\}$ be such that*

$$m = I(x) \text{ and } F(x) = F(\frac{1}{x}). \tag{1}$$

Then, the following statements hold:
i) If $m \neq \pm 1$ and $m \neq 0$ then there exists one and only one x with property (1).
ii) If $m = \pm 1$, there exist two real numbers satisfying property (1).
iii) If $m = 0$ there exist countably many real numbers x for which (1) *is true.*

Proof: Let us begin by observing that if $F(x) = 0$, then $x = I(x)$. Therefore, x fulfills (1) if and only if $x = m$ and $0 = F(m) = F(\frac{1}{m})$. Hence, m and $\frac{1}{m}$ are integers or $m = \pm 1$.

Now, we investigate those real numbers x satisfying (1) with $F(x) \neq 0$ and $m \in \mathbb{Z} \setminus \{0\}$ being fixed. We examine two cases:
i) For this case, we assume that $m \geq 1$. If x satisfies (1) than $F(x) = d_m \in [0, 1]$. Hence, by definition, $x = m + d_m$ and $\frac{1}{x} = \frac{1}{m+d_m} = d_m$. Therefore, d_m is the unique solution belonging to $[0, 1]$ of the equation $d^2 + md - 1 = 0$. If $m > 1, x$ is unique while $x = 1$ and $x = 1 + d_1$ if $m = 1$.
ii) In this case, we assume that $m \leq -1$. Then, $F(x) = d_m \in [-1, 0]$ so that $x = m + d_m$ and $\frac{1}{x} = \frac{1}{m+d_m} = d_m$. Therefore, d_m is the unique solution belonging to $[-1, 0]$ of the equation $d^2 + md - 1 = 0$. Hence, if $m < -1$, x is unique. Furthermore, if $m = -1$ then we have $x = -1$ and $x = -1 + d_{-1}$. Observe that when $m = 0$, the x values fulfilling (1) are the reciprocals of the numbers y with $I(y) \neq 0$. □

In the previous proof, we have seen that, for a fixed $m \in \mathbb{Z} \setminus \{0\}$, the $x_m \neq \pm 1$ fulfilling (1) are such that

$$F(x_m) = F(\frac{1}{x_m}) = d_m \ = \begin{cases} \frac{-m+\sqrt{m^2+4}}{2} & \text{if } m > 0, \\ \frac{-m-\sqrt{m^2+4}}{2} & \text{if } m < 0. \end{cases}$$

Obviously, $\frac{1}{x_m} = d_m$. Moreover, $x_m^2 + (\frac{1}{x_m})^2 = (m + d_m)^2 + d_m^2 = 2d_m(m + d_m) + m^2$ $= 2 + m^2$.

Hence, we have proved the following theorem.

Theorem 3.3: *For each $m \in \mathbb{Z} \setminus \{0\}$, the equation $d^2 + md - 1 = 0$ has only one solution such that the point $P_m = (m + d, d)$ is characterized by the following properties: P_m is on the equilateral hyperbola $y = \frac{1}{x}$, its coordinates are isodecimal numbers and the sum of the squares of the coordinates is equal to $2 + m^2$.*

4. NUMBERS ISODECIMAL TO THEIR SQUARE

Theorem 4.1: *Let x be a real number then:*
i) if $x > 0$ then x is isodecimal to x^2 if and only if there exists an integer $h \geq 0$ such that $x = \frac{1+\sqrt{1+4h}}{2}$;
ii) if $x \leq 0$ then x is isodecimal to x^2 if and only if x is an integer.

Proof: i) If $x > 0$ then, by applying (i) of Lemma 2.1, x is isodecimal to x^2 if and only if $x^2 - x = h$ with $h \in \mathbb{Z}$; the result follows because x is a positive integer.
ii) If $x \leq 0$ then, by (ii) of Lemma 2.1, x is isodecimal to x^2 if and only if x is an integer. In this case, $x = \frac{1-\sqrt{1+4h}}{2}$ where $h = n(n-1)$ with n a positive integer. □

5. NUMBERS ISODECIMAL TO BOTH THE RECIPROCAL AND SQUARE

Theorem 5.1: *The only real numbers $x \neq 0$ isodecimal to both x^2 and $\frac{1}{x}$ are $x = \pm 1$ and $x = \frac{1+\sqrt{5}}{2}$.*

Proof: By Theorem 3.1 and Theorem 4.1, if $F(x) = F(\frac{1}{x}) = F(x^2)$ then $x = \frac{k\pm\sqrt{k^2+4}}{2} = \frac{1\pm\sqrt{1+4h}}{2}$ with $h, k \in \mathbb{Z}$. Therefore, we have

$$k - 1 \pm \sqrt{k^2 + 4} = \pm\sqrt{1 + 4h}. \tag{2}$$

Squaring both sides of the equation (2), we obtain

$$\mp(k - 1)\sqrt{k^2 + 4} = 2h - k^2 + k - 2. \tag{3}$$

The left hand in (3) is an integer only if $k = 0$ or $k = 1$. If $k = 0$ we obtain $x = \pm 1$ satisfying $F(x) = F(x^2) = F(\frac{1}{x})$. If $k = 1$, by Theorem 3.1, we have $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ that are isodecimal to their inverse. By applying Theorem 4.1, we conclude that α is isodecimal also to α^2, while β is not isodecimal to β^2. □

6. ISODECIMAL POINTS ON EQUILATERAL HYPERBOLAS

Definition 6.1: *A point (x, y) of $\mathbb{R}^2$ is called isodecimal if x is isodecimal to y.*

Let a be a fixed real number, $a \neq 0$. In this section, we investigate those isodecimal points lying on the equilateral hyperbola $y = \frac{a}{x}$.

Theorem 6.1: *We shall look at two cases:*
i) If $a > 0$ then a point (x, y) on $y = \frac{a}{x}$ is isodecimal if and only if

$$x = \frac{k \pm \sqrt{k^2 + 4a}}{2}, \quad \textit{with } k \in \mathbb{Z}. \tag{4}$$

ii)If $a < 0$ there exist isodecimal points (x, y) on $y = \frac{a}{x}$ if and only if a is an integer. In such a case there are only a finite number of isodecimal points and they are related to the decomposition of $|a|$ into prime factors.

Proof: i) This result follows immediately by (i) of Lemma 2.1.
ii) By (ii) of Lemma 2.1, x is isodecimal to $\frac{a}{x}$ if and only if there exist $n, m \in \mathbb{Z} \setminus \{0\}$ such that $a = nm$. However, the only isodecimal points on $y = \frac{a}{x}$ with $a = mn$ are (m, n). □

Next, we observe that the tangent line to $y = \frac{a}{x}$ at $(x_0, \frac{a}{x_0})$ is

$$y(x) = \frac{a}{x_0^2}(2x_0 - x). \tag{5}$$

The line (5) intersects the coordinate axes respectively at $T_1 = (2x_0, 0)$ and $T_2 = (0, \frac{2a}{x_0})$. By (iii) of Lemma 2.1, we know that if $(x_0, \frac{a}{x_0})$ is isodecimal then the abscissa of T_1 is isodecimal to the ordinate of T_2. In such a case, the point $(2x_0, \frac{2a}{x_0})$, whose projections onto the axes are T_1 and T_2, is an isodecimal point on the equilateral hyperbola $y = \frac{4a}{x}$ (see fig.1). If we iterate this process, we see that the point $(4x_0, \frac{4a}{x_0})$ is isodecimal on the equilateral hyperbola $y = \frac{16a}{x}$. Obviously, the iteration process can be continued. Letting H_n be the equilateral hyperbola $y = \frac{4^n a}{x}, n = 0, 1, 2......$, the geometric construction illustrated above provides a one-to-one mapping τ_n of H_0 onto H_n. It is obvious that if $Q_0 = (x_0, \frac{a}{x_0})$ then $Q_n = \tau_n(Q_0) = (2^n x_0, \frac{2^n a}{x_0})$. An immediate consequence of (iii) of Lemma 2.1 is that if Q_0 is isodecimal then Q_n is isodecimal too.

Fig. 1 Construction of the application τ_1

Now, we observe that if $a > 0$ by (4) there exist $k, h \in \mathbb{Z}$ such that

$$x_0 = \frac{k \pm \sqrt{k^2 + 4a}}{2} \tag{6}$$

and

$$2^n x_0 = \frac{h \pm \sqrt{h^2 + 4^{n+1}a}}{2}. \tag{7}$$

The integer h is related to k by the identity

$$2^n k \pm \sqrt{(2^n k)^2 + 4^{n+1}a} = h \pm \sqrt{h^2 + 4^{n+1}a}. \tag{8}$$

By simple calculations, we see that (8) is true if and only if $h = 2^n k$. Hence, we obtain the following theorem:

Theorem 6.2: *Let $a > 0$ and H_n $(n = 0, 1, 2....)$ be the equilateral hyperbola $y = \frac{4^n a}{x}$. A point $P = (z_0, \frac{4^n a}{z_0}) \in H_n$ is an isodecimal point corresponding to an isodecimal point $Q \in H_0$ if and only if*

$$z_0 = \frac{h \pm \sqrt{h^2 + 4^{n+1}a}}{2}$$

with $h = 2^n k$, where k is an integer.

7. ISODECIMAL POINTS ON PARABOLA

Let a be a fixed number, $a \neq 0$. In this section, we investigate those isodecimal points $P = (x, ax^2)$ lying on the parabola $y = ax^2$.

Theorem 7.1: *We shall look at two cases:*
i)If $sgn(x) = sgn(a)$ then P is isodecimal if and only if there exists $h \in \mathbb{Z}$ such that

$$x = \frac{1 \pm \sqrt{1 + 4ah}}{2a}.$$

ii) If $sgn(x) \neq sgn(a)$, the point P is isodecimal if and only if

$$a = \frac{q}{p^2} \text{ and } x = p \text{ with } p, q \in \mathbb{Z} \setminus \{0\}.$$

Proof: i) This result follows immediately from (i) of Lemma 2.1.
ii) By (ii) of Lemma 2.1, P is isodecimal if and only if there exists $p, q \in \mathbb{Z} \setminus \{0\}$ such that $x = p,\ ax^2 = q$. □

Let $L_i,\ i = 0, 1$, be the parabola $y = (-4)^i ax^2$. The geometric process introduced in the previous section allows us to construct a one-to-one mapping $\sigma : L_0 \to L_1$.

The tangent line to L_0 at $P_0 = (x_0, ax_0^2)$ is $y(x) = ax_0(2x - x_0)$, such a line intersects the coordinates axes respectively at $S_1 = (\frac{x_0}{2}, 0)$ and $S_2 = (0, -ax_0^2)$; the point $P_1 = (\frac{x_0}{2}, -ax_0^2)$, whose projections onto the axes are S_1 and S_2, is on the parabola $y = -4ax^2$. We now define

$\sigma(x, ax^2) = (\frac{x}{2}, -ax^2)$. It is obvious that σ is a one-to-one mapping that doesn't preserves the property to be isodecimal.

Theorem 7.2: *A necessary and sufficient condition that the point* $P_0 = (x_0, ax_0^2)$, *with* $x_0 \neq 0$, *and the point* $\sigma(P_0) = (\frac{x_0}{2}, ax_0^2)$ *are both isodecimal is that*

$$a = \frac{q}{4p^2} \text{ with } p, q \in \mathbb{Z} \text{ and } x_0 = 2p. \tag{9}$$

Proof: We examine two cases:
i) For this case, we assume that $sgn(x_0) = sgn(a)$. Then, by (i) of Theorem 7.1, P_0 is isodecimal if and only if

$$x_0 = \frac{1 \pm \sqrt{1 + 4ah}}{2a}, \text{ with } h \in \mathbb{Z}. \tag{10}$$

By (ii) of Theorem 7.1, $\sigma(P_0) = (\frac{x_0}{2}, -ax_0^2)$ is isodecimal if and only if there exist $p, q \in \mathbb{Z}$, such that $\frac{x_0}{2} = p$ and $-ax_0^2 = -q$. Then

$$a = \frac{q}{4p^2} \text{ and } x_0 = 2p \text{ with } p, q \in \mathbb{Z}. \tag{11}$$

The choice of a and x_0 as in (11) fulfills (10) where $h = q - 2p$ and the sign is $+$ if $p(p-q) < 0$, while we have the same value of h and the sign $-$ if $p(p-q) > 0$.
ii) In this case, we assume $sgn(x_0) \neq sgn(a)$. By (ii) of Theorem 7.1, P_0 is isodecimal if and only if there exist $m, q \in \mathbb{Z}$ such that $x_0 = m$ and $ax_0^2 = q$. Then

$$a = \frac{q}{m^2} \text{ and } x_0 = m \text{ with } m, q \in \mathbb{Z}. \tag{12}$$

By (i) of Theorem 7.1, $\sigma(P_0) = (\frac{x_0}{2}, -ax_0^2)$ is isodecimal if and only if

$$x_0 = \frac{-1 \pm \sqrt{1 + 16ah}}{4a}, \text{ with } h \in \mathbb{Z}. \tag{13}$$

Combining (12) and (13), we see that m must be even, say $m = 2p$. Let

$$x_0 = 2p, \ a = \frac{q}{4p^2} \text{ and } h = q + p \text{ with } p, q \in \mathbb{Z}.$$

If $p(p+2q) > 0$ then (13) is fulfilled with the sign $+$; if $p(p+2q) < 0$ then (13) is fulfilled with the sign $-$.

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