THREE NEW EXTRACTION FORMULAE

Wai-Fong Chuan

Department of Applied Mathematics, Chung-Yuan Christian University
Chung Li, Taiwan 32023, R.O.C.
e-mail: wfc@math2.math.nthu.edu.tw

Fei Yu

Yuanpei University of Science and Technology, 306 Yuanpei St.,
Hsinchu City, Taiwan 30092, R.O.C.
e-mail: aypyufei@mail.yust.edu.tw

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ABSTRACT

Let α be an irrational number between 0 and 1. Let a and b be distinct letters. Define $d_n = a$ (resp., b) if $[(n+1)\alpha] - [n\alpha] = 0$ (resp., 1), $n \in \mathbb{Z}$. Define x to be the two-way infinite word whose n^{th} letter is $d_n, n \in \mathbb{Z}$. Define $x_m = d_{m+1}d_{m+2}\cdots, m \in \mathbb{Z}$, $s_0 = \varepsilon$, the empty word, $s_m = d_1 d_2 \cdots d_m, m \geq 1$. The problem of determining the extracted word $\langle x_m, x_0 \rangle$ obtained by aligning x_m with x_0 was originally posed by D.R. Hofstadter in 1963. Known extraction formulae include $\langle x_m, x_0 \rangle$ $(m > 0)$ (by R.J. Hendel and S.A. Monteferrante 1994), $\langle x_0, x_m \rangle$ $(m \geq 1)$ (by W. Chuan 1995) for $\alpha = (\sqrt{5}-1)/2$ and partial results for $\langle x_m, x_0 \rangle$ $(m \geq 1)$ (by R.J. Hendel 1996) and all cases of $\langle x_0, x_m \rangle$ $(m \geq 0)$ (by W. Chuan and F. Yu 2000) for $\alpha = \sqrt{2}-1$. In this short note, we establish the following three new extraction formulae for $\alpha = (\sqrt{5}-1)/2$:

$$\langle x_m, x_{-2} \rangle = x_m \ (m > -2)$$
$$\langle x_m, x_{-2} \rangle = R(s_{-m-2}) \ (m \leq -2)$$

$$\langle x_0, x_{-m} \rangle = \begin{cases} x_{m-2} \ (m > 1) \\ bx_0 \neq x_{-1} \ (m = 1) \end{cases}$$

which involve x_m, where $m < 0$. We also show that the first formula is equivalent to the formula proved by Hendel and Monteferrante.

1. INTRODUCTION

Throughout this paper, we consider only words over the alphabet $\{a, b\}$ and we adopt notations from [3,6,7,8]. Let ε denote the empty word. For any word $w = a_1 a_2 \cdots a_n$, where $n \geq 1$, $a_i \in \{a, b\}$, $1 \leq i \leq n$, define the reversal $R(w)$ and the length $|w|$ of w by $R(w) = a_n \cdots a_2 a_1$, $|w| = n$, $R(\varepsilon) = \varepsilon$, and $|\varepsilon| = 0$. A word w is said to be a *palindrome* if $R(w) = w$. If $w, w_1, w_2, \cdots$ are words, products, powers are defined as usual by $w^0 = \varepsilon$, $w^1 = w$, $w^{n+1} = w^n w, n \geq 2$, $\prod_{i=1}^{\infty} w_i = w_1 \prod_{i=2}^{\infty} w_i$. A nonempty word u is said to be a *prefix* (resp., *suffix*) of w if there exists a nonempty word x such that $w = ux$ (resp., $w = xu$).

Let α be an irrational number between 0 and 1. Define $d_n = a$ (resp., b) if $[(n+1)\alpha] - [n\alpha] = 0$ (resp., 1), $n \in \mathbb{Z}$. Define $x = x(\alpha)$ to be the two-way infinite word whose n^{th} letter is

$d_n,\ n \in \mathbb{Z}$. Define $s_0 = \varepsilon,\ s_m = d_1 d_2 \cdots d_m,\ m \geq 1,\ x_m = d_{m+1} d_{m+2} \cdots,\ m \in \mathbb{Z}$. Each x_m is called a *suffix* of x. x_0 is called the *characteristic word* of α. Clearly, $x_0 = s_m x_m,\ m \geq 0$. For $\alpha = (\sqrt{5}-1)/2$, the word x_0 (resp., x) is the *golden sequence* (resp., *two-way infinite golden sequence*) (see [11]). x_0 is also called the *infinite Fibonacci word*.

Originally, Hofstadter [9] formulated the concept of aligning x_m with $x_0,\ m \geq 1$ (see also [3,6,7,8]). The idea is to try to match each term (letter) in x_0 with a term in x_m, beginning at the first term of x_m. After a term in x_0 has been matched with a term in x_m, one looks for the earliest match to the next term in x_0. Those terms in x_m that are skipped over from the extracted word $\langle x_m, x_0 \rangle$. For example, when $\alpha = (\sqrt{5}-1)/2$ and $m = 4$,

$$\begin{array}{rcccccccccccccccccccccc}
x_m & = & a & b & a & b & b & a & b & b & a & b & a & b & b & a & b & a & b & b & a & \cdots \\
x_0 & = &   & b & a & b & b & a & b &   & a & b &   & b &   & a & b &   & b &   & a & \cdots \\
\langle x_m, x_0 \rangle & = & a & & & & & & & b & & & a & & b & & & a & & b & & \cdots
\end{array} \tag{1.1}$$

Here we say that x_m *aligns* (with) x_0 with *extraction* $\langle x_m, x_0 \rangle$. The word x_0 is called the *aligned word*. The relationship (1.1) is an *alignment*. Hendel and Monteferrante [8] were the first to provide a rigorous definition of alignment of finite words. Hendel [7] was the first to introduce the functional notation $\langle x_m, x_0 \rangle$. The original notation for $\langle u, v \rangle = w$ was $u \supset v;\ w$. In [9], Hofstadter conjectured that $\langle x_m, x_0 \rangle = x_{m-2}$, for $m \geq 2$. Hendel and Monteferrante [8] observed that this was not always the case, and for $\alpha = (\sqrt{5}-1)/2$, they successfully established a modified formula for $\langle x_m, x_0 \rangle$. In order to state their result, we need to define the notation m^*.

Lemma A:

(a) (see [2,10]) Each positive integer m has a unique representation as $m = \sum_{i=1}^{n} r_i F_{i+1}$, where

$$r_i \in \{0,1\},\ r_i + r_{i+1} \geq 1,\ 1 \leq i \leq n-1, \ \text{ and } \ r_n = 1. \tag{1.2}$$

(This representation of m is called the *maximal representation* of m.)

(b) (see [1,10]) Each positive integer m can be expressed uniquely as $m = \sum_{i=1}^{n} r_i F_{i+1}$, where $r_n = 1,\ r_i \in \{0,1\}$, and $r_i = 0$ whenever $r_{i+1} = 1,\ 1 \leq i \leq n-1$. (This result is known as Zeckendorf's theorem, and this representation of m is called the *minimal representation* or *Zeckendorf representation* of m.)

If m is a positive integer and $m = \sum_{i=1}^{n} r_i F_{i+1}$ is the minimal representation of m given by part (b) of Lemma A, define a binary string $m^* = r_1 r_2 \cdots r_n$. Define $0^* = \lambda$, the empty binary string. Let

$$\begin{aligned} M = \{m \in \mathbb{Z}_+ : m^* = &10^{2k-1}1s \ \text{ for some } \ k \in \mathbb{Z}_+ \\ &\text{and some binary string } s\}. \end{aligned} \tag{1.3}$$

The modified formula for $\langle x_m, x_0 \rangle$, proved by Hendel and Monteferrante [8] for $\alpha = (\sqrt{5}-1)/2$ is as follows.

Theorem B: For $m \geq 2$,

$$\langle x_m, x_0 \rangle = \begin{cases} x_{m-2}, & \text{if } \ m \notin M, \\ a x_{m-1} \neq x_{m-2}, & \text{if } \ m \in M. \end{cases} \tag{1.4}$$

The extractions $\langle x_0, x_n\rangle$ and $\langle x_m, x_n\rangle$, where m, $n \geq 1$, were first considered by Chuan [3] who proved the following formula for $\alpha = (\sqrt{5}-1)/2$.

Theorem C: $\langle x_0, x_n\rangle = R(s_n),\ n \geq 1.$ (1.5)

In [3], Chuan also proved that

$$\langle x_m, x_n\rangle \text{ differs from } x_{m-n-2} \text{ (if } m > n \geq 0) \text{ or from } R(s_{n-m}) \text{ (if } n > m \geq 0) \text{ by at most the first letter.} \tag{1.6}$$

For $\alpha = \sqrt{2}-1$, Hendel proved some results for $\langle x_m, x_0\rangle$ and $\langle x_0, x_m\rangle$, $m \geq 1$ (see[7]). Chuan and Yu introduced the subtraction rule for exponents, which is equivalent to the equation $\langle x_0, x_m\rangle = R(s_m),\ m \geq 0$ (see [6]). In this short note, we extend the extraction problem for $\alpha = (\sqrt{5}-1)/2$ to include x_m, where $m < 0$.

The new extraction formulae are

Theorem 1.1: $\langle x_m, x_{-2}\rangle = x_m,\ m > -2.$ (1.7)

Theorem 1.2: $\langle x_m, x_{-2}\rangle = R(s_{-m-2})$, for $m \leq -2$. (1.8)

Theorem 1.3: $\langle x_0, x_{-m}\rangle = x_{m-2},\ m \geq 2,$ (1.9)

$\langle x_0, x_{-1}\rangle = bx_0 \neq x_{-1}.$ (1.10)

We remark that Theorem 1.3 directly extends Theorem C; Theorem 1.2 clearly extends Theorem B to negative m; in Theorem 3.4 below, we show that Theorem B and Theorem 1.1 are equivalent. It is remarkable that the extracted words obtained in Theorem 1.1 and 1.2 are always suffixes and reversals of prefixes of x respectively. The methods used in this paper, can be used to generalize Theorems 1.1-1.3 to the case $\alpha = \sqrt{2}-1$.

We first state some known results that will be used later. Define a sequence $\{w_n\}$ of words by

$$w_1 = a,\ w_2 = b,\ w_n = w_{n-2}w_{n-1}\ (n \geq 3).$$

Clearly

$$|w_n| = F_n, \quad \text{for} \quad n \geq 1. \tag{1.11}$$

Lemma D:

(a) (see Lemma 3.10 and Corollary 3.8 of [5], [8]) Let $m \geq 0$. If $m = \sum_{i=1}^{n} r_i F_{i+1}$ where $r_i \in \{0,1\}$ $(1 \leq i \leq n)$, then

$$R(s_m) = w_2^{r_1} w_3^{r_2} \cdots w_{n+1}^{r_n}, \tag{1.12}$$

$$x_m = w_2^{1-r_1} w_3^{1-r_2} \cdots w_{n+1}^{1-r_n} w_{n+2} w_{n+3} \cdots . \tag{1.13}$$

(b) (see [3])

$$w_n = w_2(w_1 w_2 \cdots w_{n-2}), \quad \text{if} \quad n \geq 4 \text{ is even.} \tag{1.14}$$

$$s_m = R(w_n) s_{m-F_n}, \text{ if } F_n \leq m \leq F_{n+2}-2,\ n \geq 2. \tag{1.15}$$

(c) (see [8]) If u_n, v_n and e_n are words with $\langle u_n, v_n\rangle = e_n,\ n = 1, 2\cdots$, then $\langle \prod u_n, \prod v_n\rangle = \prod e_n$.

(d) (see [8])

$$\langle w_n, w_{n-1}\rangle = w_{n-2} \text{ for } n \geq 3. \tag{1.16}$$

2. PROOFS OF THE MAIN THEOREMS

In order to prove the main theorems, we first use the known factorizations (1.12)-(1.14) of $R(s_m)$ and x_m to derive more factorizations of suffixes of x in terms of w_n's.

Lemma 2.1:

(a) $d_{-n} = d_{n-1}$ $(n \geq 2)$. (2.1)

(b) $x_{-2} = w_{2n}w_{2n-1}w_{2n}w_{2n+1}\cdots$ $(n \geq 1)$. (2.2)

(c) $x_{-m} = R(s_{m-2})x_{-2}$ $(m \geq 2)$. (2.3)

(d) Let $m \geq 0$. Let $n \geq 0$ be such that $F_{n+2} - 1 \leq m \leq F_{n+3} - 2$. Then

$$x_m = R(s_k)w_{n+3}w_{n+4}\cdots, \text{ where } k = F_{n+4} - m - 2. \tag{2.4}$$

Proof: Part (a) is clear. Part (b) follows from (1.13) with $m = 0$, and (1.14). Part (c) follows from (2.1).

(d): The case $m = 0$ is trivial. Now let $m \geq 1$. Since $F_{n+2} - 1 \leq m \leq F_{n+3} - 2$, $m = \sum_{i=1}^{n} r_i F_{i+1}$, for some $r_i \in \{0, 1\}$ $(1 \leq i \leq n)$.

Clearly,

$$F_{n+4} - 2 - m = \sum_{i=1}^{n+1} F_{i+1} - \sum_{i=1}^{n} r_i F_{i+1} = \sum_{i=1}^{n}(1 - r_i)F_{i+1} + F_{n+2}.$$

Therefore, by (1.12),

$$R(s_k) = w_2^{1-r_1}w_3^{1-r_2}\cdots w_{n+1}^{1-r_n}w_{n+2},$$

where $k = F_{n+4} - 2 - m$. Consequently, by (1.13),

$$x_m = w_2^{1-r_1}w_3^{1-r_2}\cdots w_{n+1}^{1-r_n}w_{n+2}w_{n+3}\cdots = R(s_k)w_{n+3}w_{n+4}\cdots. \quad \square$$

Lemma 2.2: $\langle x_{-2}, w_n w_{n+1}\cdots\rangle = w_{n+1}$, for $n \geq 1$. (2.5)

Proof: We repeatedly apply Lemma D (c) to the representation (2.2) of x_{-2}. If $n = 1$, then

$$\begin{aligned} \langle x_{-2}, w_1w_2\cdots\rangle &= \langle w_2(w_1w_2\cdots), w_1w_2\cdots\rangle \\ &= \langle w_2w_1, w_1\rangle\langle w_2w_3\cdots, w_2w_3\cdots\rangle \\ &= w_2. \end{aligned}$$

If $n \geq 3$ is odd, then

$$\begin{aligned} &\langle x_{-2}, w_nw_{n+1}\cdots\rangle = \langle w_{n+1}(w_nw_{n+1}\cdots), w_nw_{n+1}\cdots\rangle \\ &= \langle w_{n+1}, w_n\rangle\langle w_nw_{n+1}, w_{n+1}\rangle\langle w_{n+2}w_{n+3}\cdots, w_{n+2}w_{n+3}\cdots\rangle \\ &= w_{n-1}w_n \text{ (by (1.16))} \\ &= w_{n+1}. \end{aligned}$$

If n is even, then

$$\begin{aligned}
&\langle x_{-2}, w_n w_{n+1}\cdots\rangle = \langle w_n(w_{n-1}w_n\cdots), w_n w_{n+1}\cdots\rangle \\
&= \langle w_n, w_n\rangle\langle w_{n-1}w_n, w_{n+1}\rangle\langle w_{n+1}w_{n+2}, w_{n+2}\rangle\langle w_{n+3}w_{n+4}\cdots, w_{n+3}w_{n+4}\cdots\rangle \\
&= w_{n+1} \text{ (by (1.16)).} \quad \square
\end{aligned}$$

Lemma 2.3: Let $m \geq 3$. Let $n \geq 2$ be such that either $F_{n+2} \leq m \leq F_{n+3}$ and n is even, or $F_{n+2}+1 \leq m \leq F_{n+3}-1$ and n is odd. Then

$$\langle R(s_m), w_2(w_1w_2\cdots w_n)\rangle = R(s_{m-F_{n+2}}). \tag{2.6}$$

Proof: We proceed by induction on n. When $n = 2$ or 3, the result clearly holds. Suppose that $k \geq 3$ and that the result holds for all $n \leq k$. Now let $n = k+1$. Let $F_{k+3} \leq m \leq F_{k+4}-1$. There are five cases to consider:

Case 1: $m = F_{k+3}$;

Case 2: $m = 2F_{k+2}$;

Case 3: $F_{k+3}+1 \leq m \leq 2F_{k+2}-1$;

Case 4: $2F_{k+2}+1 \leq m \leq F_{k+4}-2$;

Case 5: $m = F_{k+4}-1$.

We prove only Cases 2 and 4. The proof of Case 1 (resp., Cases 3 and 5) is similar to Case 2 (resp., Case 4).

Proof of Case 2. $m = 2F_{k+2}$:

$$\begin{aligned}
&\langle R(s_m), w_2(w_1w_2\cdots w_{k+1})\rangle \\
&= \begin{cases} \langle w_{k+2}w_{k+2},\ w_{k+3}\rangle & \text{if k is even} \\ \langle w_{k+2}w_{k+2},\ w_{k+2}w_{k+1}\rangle & \text{if k is odd} \end{cases} \qquad \text{(by (1.11), (1.15), (1.14))} \\
&= w_k \text{ (by (1.16)).}
\end{aligned}$$

Proof of Case 4. $2F_{k+2}+1 \leq m \leq F_{k+4}-2$: Since $F_{k+2}+1 \leq m - F_{k+2} \leq F_{k+3}-2$, it follows that

$$\begin{aligned}
&\langle R(s_m), w_2(w_1w_2\cdots w_{k+1})\rangle \\
&= \langle R(s_{m-F_{k+2}}), w_2(w_1w_2\cdots w_k)\rangle\langle w_{k+2}, w_{k+1}\rangle \text{ (by (1.15))} \\
&= R(s_{m-F_{k+2}-F_{k+2}})w_k \text{ (by the inductive hypothesis and (1.16))} \\
&= R(s_{m-2F_{k+2}+F_k}) \text{ (by (1.15))} \\
&= R(s_{m-F_{k+3}}).
\end{aligned}$$

Therefore the result holds for $n = k+1$. This completes the proof. $\square$

Proof of Theorem 1.1: We consider $m \geq 3$. Let $F_{n+2}-1 \leq m \leq F_{n+3}-2$, where $n \geq 2$. Let $k = F_{n+4}-m-2$. Then $F_{n+2} \leq k \leq F_{n+3}-1$. There are two cases.

Case 1. $k = F_{n+2}$ **and** n **is odd**:

$$\begin{aligned}
&\langle x_m,\ x_{-2}\rangle \\
&= \langle R(s_k),\ w_2(w_1w_2\cdots w_{n-1})\rangle\langle w_{n+3}w_{n+4}\cdots, w_nw_{n+1}\cdots\rangle \quad \text{(by (2.2), (2.4))} \\
&= R(s_{k-F_{n+1}})\langle w_{n+3}, w_nw_{n+1}\rangle\langle w_{n+4}, w_{n+2}w_{n+3}\rangle \prod_{i=n+4}^{\infty}\langle w_{i+1}, w_i\rangle \quad \text{(by Lemma 2.3)} \\
&= R(s_{k-F_{n+1}})w_{n+1}w_{n+3}w_{n+4}\cdots \ \text{(by (1.16))} \\
&= R(s_k)w_{n+3}w_{n+4}\cdots \ \text{(by (1.15))} \\
&= x_m \ \text{(by (2.4))}.
\end{aligned}$$

Case 2. Either $F_{n+2} \le k \le F_{n+3} - 1$ **and** n **is even, or** $F_{n+2} + 1 \le k \le F_{n+3} - 1$ **and** n **is odd**:

$$\begin{aligned}
&\langle x_m,\ x_{-2}\rangle \\
&= \langle R(s_k),\ w_2(w_1w_2\cdots w_n)\rangle\langle w_{n+3}w_{n+4}\cdots, w_{n+1}w_{n+2}\cdots\rangle \quad \text{(by (2.2), (2.4))} \\
&= R(s_{k-F_{n+2}})\langle w_{n+3}, w_{n+1}w_{n+2}\rangle \prod_{i=n+3}^{\infty}\langle w_{i+1}, w_i\rangle \ \text{(by Lemma 2.3)} \\
&= R(s_{k-F_{n+2}})w_{n+2}w_{n+3}\cdots \ \text{(by (1.16))} \\
&= R(s_k)w_{n+3}w_{n+4}\cdots \ \text{(by (1.15))} \\
&= x_m \ \text{(by (2.4))}.
\end{aligned}$$

The proofs for $m = -1, 0, 1, 2$ are almost identical to the above proof. □

Proof of Theorem 1.2: We consider $m \ge 6$. Let $n \ge 2$ be such that either $F_{n+2} \le m - 2 \le F_{n+3}$ and n is even, or $F_{n+2} + 1 \le m - 2 \le F_{n+3} - 1$ and n is odd. Then

$$\begin{aligned}
&\langle x_{-m}, x_{-2}\rangle \\
&= \langle R(s_{m-2}),\ w_2(w_1w_2\cdots w_n)\rangle\langle x_{-2}, w_{n+1}w_{n+2}\cdots\rangle \\
&\qquad \text{(by (2.2), (2.3))} \\
&= R(s_{m-2-F_{n+2}})w_{n+2} \ \text{(by Lemma 2.3 and (2.5))} \\
&= R(s_{m-2}) \ \text{(by (1.15))}.
\end{aligned}$$

The proof for $m = 2, 3, 4, 5$ is almost identical to the above proof. □

Finally, we use the following lemma to prove Theorem 1.3 (see [6] for a similar lemma for the case $\alpha = \sqrt{2} - 1$).

Lemma 2.4 (Subtraction rule of exponents): Let $n \ge 1$. If $r_1r_2\cdots r_n$ is a string such that (1.2) holds then

$$\langle w_2w_3\cdots w_{n+1},\ w_2^{r_1}w_3^{r_2}\cdots w_{n+1}^{r_n}\rangle = w_2^{1-r_1}w_3^{1-r_2}\cdots w_{n+1}^{1-r_n}. \tag{2.7}$$

Proof: We proceed by induction on n. When $n = 1, 2, 3, 4$, the result clearly holds. Suppose that $k \ge 4$ and that the result holds for $n \le k$. Now let $n = k + 1$. Let $r_1r_2\cdots r_n$ be a string satisfying (1.2). There are two cases to consider:

Case 1: $r_1 r_2 \cdots r_{k+1} = r_1 r_2 \cdots r_{k-1} 11$:

$$\begin{aligned}
&\langle w_2 w_3 \cdots w_{k+2},\ w_2^{r_1} w_3^{r_2} \cdots w_{k+2}^{r_{k+1}} \rangle \\
&= \langle w_2 w_3 \cdots w_{k+1},\ w_2^{r_1} w_3^{r_2} \cdots w_{k+1}^{r_k} \rangle \langle w_{k+2}, w_{k+2} \rangle \\
&= w_2^{1-r_1} w_3^{1-r_2} \cdots w_{k+1}^{1-r_k} \text{ (by the inductive hypothesis)} \\
&= w_2^{1-r_1} w_3^{1-r_2} \cdots w_{k+2}^{1-r_{k+1}}.
\end{aligned}$$

Case 2: $r_1 r_2 \cdots r_{k+1} = r_1 r_2 \cdots r_{k-2} 101$:

$$\begin{aligned}
&\langle w_2 w_3 \cdots w_{k+2},\ w_2^{r_1} w_3^{r_2} \cdots w_{k+2}^{r_{k+1}} \rangle \\
&= \langle w_2 w_3 \cdots w_k,\ w_2^{r_1} w_3^{r_2} \cdots w_k^{r_{k-1}} \rangle \langle w_{k+1} w_{k+2}, w_{k+2} \rangle \\
&= w_2^{1-r_1} w_3^{1-r_2} \cdots w_k^{1-r_{k-1}} w_{k+1} \\
&\qquad \text{(by the inductive hypothesis and (1.16))} \\
&= w_2^{1-r_1} w_3^{1-r_2} \cdots w_{k+2}^{1-r_{k+1}}.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3: Proof of (1.9): When $m = 2$, (1.9) follows from (1.7). Now let $m > 2$, and let $m - 2 = \sum_{i=1}^{n} r_i F_{i+1}$ be the maximal representation of $m - 2$ given by part (a) of Lemma A. Then

$$\begin{aligned}
&\langle x_0,\ x_{-m} \rangle \\
&= \langle w_2 w_3 \cdots w_{n+1},\ R(s_{m-2}) \rangle \langle w_{n+2} w_{n+3} \cdots, x_{-2} \rangle \text{ (by (1.13), (2.3))} \\
&= \langle w_2 w_3 \cdots w_{n+1},\ w_2^{r_1} w_3^{r_2} \cdots w_{n+1}^{r_n} \rangle \langle w_{n+2} w_{n+3} \cdots, x_{-2} \rangle \text{ (by (1.12))} \\
&= w_2^{1-r_1} w_3^{1-r_2} \cdots w_{n+1}^{1-r_n} w_{n+2} w_{n+3} \cdots \text{ (by (2.7), (1.7))} \\
&= x_{m-2} \text{ (by (1.13))}.
\end{aligned}$$

This proves (1.9).

Proof of (1.10):

$$\langle x_0, x_{-1} \rangle = \langle bab, ab \rangle \prod_{i=3}^{\infty} \langle w_{i+1}, w_i \rangle = b w_2 w_3 \cdots = b x_0 \neq x_{-1}. \quad \square$$

3. EQUIVALENCE OF THEOREM B AND THEOREM 1.1

In this section, we show that Theorem B and Theorem 1.1 are equivalent.

Lemma 3.1 (see Theorem 3.1 of [4]): Let $m \geq 0$. Then the prefix of x_m having length 2 is bb if and only if $m^* = 01s$ for some binary string s.

Lemma 3.2: Let M be the set defined by (1.3). Then

$$M = \{m \in \mathbb{Z}_+ : x_{m-2} = bbx_m\}. \tag{3.1}$$

Proof: Since the sets on both sides of (3.1) do not contain 1, we consider only $m \geq 2$. Applying Lemma 3.1 with $m-2$ in place of m, we see that

$$\begin{aligned}
&\text{the prefix of } x_{m-2} \text{ having length 2 is } bb\\
&\Leftrightarrow (m-2)^* = 01s \text{ for some binary string } s\\
&\Leftrightarrow m^* = 10^{2k-1}1s' \text{ for some } k \in \mathbb{Z}_+ \text{ and some binary string } s'\\
&\Leftrightarrow m \in M. \quad \square
\end{aligned}$$

Lemma 3.3 (see, for example, Theorem 3.1 of [4]): The words aa, bbb and $ababa$ are not factors of x.

Theorem 3.4: Theorem B and Theorem 1.1 are equivalent.

Proof: We prove that (1.4)⇔(1.7).

Proof of (1.4)⇒(1.7): Suppose that (1.4) holds. Let $m \geq -1$. By Lemma 3.3, there are four cases to consider.

Case 1: $x_m = bax_{m+2}$: By (3.1), $m+2 \notin M$. Therefore, by (1.4), $\langle x_{m+2}, x_0\rangle = x_m$. Hence $\langle x_m, x_{-2}\rangle = \langle bax_{m+2}, bax_0\rangle = \langle ba, ba\rangle\langle x_{m+2}, x_0\rangle = x_m$.

Case 2: $x_m = abax_{m+3}$: By (3.1) and (1.4), $\langle x_{m+3}, x_0\rangle = x_{m+1}$. Hence $\langle x_m, x_{-2}\rangle = \langle abax_{m+3}, bax_0\rangle = \langle aba, ba\rangle\langle x_{m+3}, x_0\rangle = ax_{m+1} = x_m$.

Case 3: $x_m = abbax_{m+4}$: By (3.1) and (1.4), $\langle x_{m+4}, x_0\rangle = x_{m+2}$. Hence $\langle x_m, x_{-2}\rangle = \langle abbax_{m+4}, bax_0\rangle = \langle abba, ba\rangle\langle x_{m+4}, x_0\rangle = abx_{m+2} = x_m$.

Case 4: $x_m = bbax_{m+3}$: By (3.1) and (1.4), $\langle x_{m+3}, x_0\rangle = x_{m+1}$. Hence $\langle x_m, x_{-2}\rangle = \langle bbax_{m+3}, bax_0\rangle = \langle bba, ba\rangle\langle x_{m+3}, x_0\rangle = bx_{m+1} = x_m$.

This proves (1.7).

Proof of (1.7) ⇒ (1.4): Suppose that (1.7) holds. Let $m \geq 2$. By Lemma 3.3, there are four cases to consider.

Case 1: $m \notin M$ **and** $x_{m-2} = bax_m$: By (1.7), $\langle x_{m-2}, x_{-2}\rangle = x_{m-2}$. Hence $\langle x_m, x_0\rangle = \langle bax_m, bax_0\rangle = \langle x_{m-2}, x_{-2}\rangle = x_{m-2}$.

Case 2: $m \notin M$ **and** $x_{m-2} = abax_{m+1} = ababx_{m+2}$: By (1.7), $\langle x_{m-2}, x_{-2}\rangle = x_{m-2}$. Hence

$$\begin{aligned}
\langle x_m, x_0\rangle &= \langle abx_{m+2}, bx_1\rangle = a\langle x_{m+2}, x_1\rangle = \langle aba, ba\rangle\langle bx_{m+2}, bx_1\rangle\\
&= \langle ababx_{m+2}, babx_1\rangle = \langle x_{m-2}, x_{-2}\rangle = x_{m-2}.
\end{aligned}$$

Case 3. $m \notin M$ **and** $x_{m-2} = abbabax_{m+4}$: By Lemma 3.3, $ababa$ is not a factor of x. Hence $x_m = bababbax_{m+7}$. Since $x_m = bax_{m+2}$, it follows from Case 1 that $\langle x_{m+2}, x_0\rangle = x_m$. Thus

$$\begin{aligned}
\langle x_m, x_0\rangle &= \langle bababbax_{m+7}, babbax_5\rangle = ab\langle x_{m+7}, x_5\rangle\\
&= ab\langle babbax_{m+7}, babbax_5\rangle = ab\langle x_{m+2}, x_0\rangle\\
&= abx_m = x_{m-2}.
\end{aligned}$$

Case 4. $m \notin M$ **and** $x_{m-2} = abbabbx_{m+4} = abbabbabx_{m+6}$: Since $x_{m-3} = bax_{m-1}$, it follows from Case 1 that $\langle x_{m-1}, x_0\rangle = x_{m-3}$. Hence

$$\begin{aligned}
b\langle x_{m+2}, x_2\rangle &= \langle bba, ba\rangle\langle x_{m+2}, x_2\rangle = \langle bbax_{m+2}, bax_2\rangle\\
&= \langle x_{m-1}, x_0\rangle = x_{m-3} = bx_{m-2}.
\end{aligned}$$

Thus $\langle x_m, x_0\rangle = \langle bax_{m+2}, bax_2\rangle = \langle x_{m+2}, x_2\rangle = x_{m-2}$.

Case 5. $m \in M$, **i.e.,** $x_{m-2} = bbx_m$: Since $x_{m-1} = bax_{m+1}$, it follows from Case 1 that $\langle x_{m+1}, x_0\rangle = x_{m-1}$. Hence

$$\begin{aligned}\langle x_m, x_0\rangle &= \langle abx_{m+2}, bx_1\rangle = a\langle bx_{m+2}, bx_1\rangle = a\langle x_{m+1}, x_0\rangle \\ &= ax_{m-1} \neq x_{m-2}.\end{aligned}$$

This proves (1.4). □

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