# FIBONACCI FRACTIONS FROM HERON'S SQUARE ROOT APPROXIMATION OF THE GOLDEN RATIO

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## ABSTRACT

Heron's method is used to approximate $\sqrt{5}$ in order to find successive rational approximations of the Golden Ratio, and a characterization is given for when the results always will be ratios of successive Fibonacci numbers.

## 1. INTRODUCTION

Throughout history, mathematicians have sought rational approximations of irrational numbers. Today, many of these approximations can be found with quickly converging infinite series; but, historically, many estimates necessarily relied first on algebraic approximations of square roots. Using a 96-sided polygon, Archimedes found the equivalent of

$$\pi \approx 48\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}.$$

Then by estimating the radicals, he found that $\pi \approx 211875/67441$ [2]. Digit seekers continued with Archimedes' method through the 17th century with Ludolph van Ceulen ultimately finding $\pi$ to 35 decimal places by using polygons with $2^{62}$ sides. Even Newton's approximation of $\pi$, which used his integral calculus, still relied on the generalized binomial theorem to approximate square roots with an infinite series [2]. When approximating the Golden Ratio though, no such problems arise because we may simply take the ratio of any two successive Fibonacci numbers $F_{n+1}/F_n$ to obtain a rational approximation.

But suppose we were to seek a rational approximation of $\Phi = (1+\sqrt{5})/2$ that first relied on an historical method of approximating $\sqrt{5}$. Would we obtain a recognizable pattern of fractions? In this article, we shall use Heron's method of approximating $\sqrt{5}$ to find successive rational approximations of $\Phi$, and give a characterization of when the iterations always will yield ratios of successive Fibonacci numbers.

## 2. HERON'S METHOD

In the first century A.D., Heron of Alexandria described a method for approximating $\sqrt{a}$, although the process may have been known much earlier. He simply let $a_0$ be an initial estimate of $\sqrt{a}$. Then for $n \geq 0$, he let

$$a_{n+1} = \frac{a_n + a/a_n}{2}$$

to obtain better estimates. By approximating $\sqrt{5}$ in this manner, we can approximate $\Phi$ by $\Phi_n = (1+a_n)/2$. With some careful choices of $a_0$, the resulting fractions will always be ratios of successive Fibonacci numbers.

For instance, using $a_0 = 2$ as the first estimate of $\sqrt{5}$, the initial approximation of $\Phi$ is given by

$$\Phi_0 = \frac{1+a_0}{2} = \frac{3}{2} = \frac{F_4}{F_3}.$$

The first iteration of Heron's method with $a_0 = 2$ yields $a_1 = 9/4$, and then

$$\Phi_1 = \frac{1+9/4}{2} = \frac{13}{8} = \frac{F_7}{F_6}.$$

Continuing, we observe that

$$\Phi_2 = \frac{F_{13}}{F_{12}},\ \Phi_3 = \frac{F_{25}}{F_{24}},\ \Phi_4 = \frac{F_{49}}{F_{48}},\ \Phi_5 = \frac{F_{97}}{F_{96}},\ \ldots$$

We are then led to conjecture:

**Proposition**: Let $a_0 = 2$ be an initial estimate of $\sqrt{5}$. For $n \geq 0$, let $a_{n+1} = (a_n + 5/a_n)/2$ be successive estimates of $\sqrt{5}$ and let $\Phi_n = (1 + a_n)/2$ be the $n$th iterative approximation of the Golden Ratio $\Phi = (1+\sqrt{5})/2$. Then $\Phi_n$ is the ratio of successive Fibonacci numbers. Specifically, $\Phi_n = F_{3\cdot 2^n+1}/F_{3\cdot 2^n}$.

## 3. OTHER PATTERNS

Interestingly enough, other initial estimates of $\sqrt{5}$ give similar results:

Let $a_0 = 3$. Then for $n \geq 0$,

$$\Phi_n = \frac{F_{2\cdot 2^n+1}}{F_{2\cdot 2^n}}. \tag{1}$$

Let $a_0 = 5/2$. Then for $n \geq 1$,

$$\Phi_n = \frac{F_{3\cdot 2^n+1}}{F_{3\cdot 2^n}}. \tag{2}$$

Let $a_0 = 7/3$. Then for $n \geq 0$,

$$\Phi_n = \frac{F_{4\cdot 2^n+1}}{F_{4\cdot 2^n}}. \tag{3}$$

Let $a_0 = 15/7$. Then for $n \geq 1$,

$$\Phi_n = \frac{F_{4\cdot 2^n+1}}{F_{4\cdot 2^n}}. \tag{4}$$

Let $a_0 = 11/5$. Then for $n \geq 0$,

$$\Phi_n = \frac{F_{5\cdot 2^n+1}}{F_{5\cdot 2^n}}. \tag{5}$$

The proofs of these results and of the Proposition can be handled individually by induction; however, we shall give a single inductive argument that handles many cases. We do note though that we may not always obtain such results as the pattern seems to fail with $a_0 = 8/3$, $11/4$, and $12/5$. So we ask the question: "What conditions on a reduced fraction $a_0 = c/d$ will result in Fibonacci fractions when applying Heron's method on $\sqrt{5}$ to obtain $a_{n+1}$ and letting $\Phi_n = (1 + a_n)/2$? Moreover, what is the resulting form?"

Our characterization is stated next:

**Theorem**: Let $a_0 = c/d$ be an initial estimate of $\sqrt{5}$ with $\gcd(c, d) = 1$. Let $a_{n+1} = (a_n + 5/a_n)/2$ and $\Phi_n = (1 + a_n)/2$. Then $\Phi_n = F_{k2^n+1}/F_{k2^n}$ for all $n \geq 0$ if and only if $c$ and $d$ satisfy either

(i) $d = F_k$ is an odd Fibonacci number and $c = 2F_{k+1} - d$, or
(ii) $F_k$ is an even Fibonacci number, $d = F_k/2$, and $c = F_{k+1} - d$.

To prove the theorem, we will need the following two Fibonacci identities credited to Lucas in 1876:

$$F_m\left(F_{m+1} + F_{m-1}\right) = F_{2m} \tag{6}$$

$$\left(F_{m+1}\right)^2 + \left(F_m\right)^2 = F_{2m+1}. \tag{7}$$

Proofs of these and many other identities can be found in [1].

**Proof of Theorem**: Suppose first that either Condition (i) or Condition (ii) is satisfied and suppose $j$ divides both $c$ and $d$. In Case (i), $j$ must be odd because $d = F_k$ is odd. But then $j$ divides $c+d = 2F_{k+1}$, so $j$ must divide $F_{k+1}$. Because the successive Fibonacci numbers $F_k$ and $F_{k+1}$ are relatively prime, $j = 1$ and thus $\gcd(c, d) = 1$. In Case (ii), if $j$ divides both $c$ and $d$, then $j$ divides $c + d = F_{k+1}$ and $j$ divides $2d = F_k$. Again, we have that $j = 1$ and $\gcd(c, d) = 1$.

In either case, we have

$$a_0 = \frac{c}{d} = \frac{2F_{k+1}}{F_k} - 1$$

and

$$\Phi_0 = \frac{1 + a_0}{2} = \frac{1}{2} + \frac{1}{2}\left(\frac{2F_{k+1}}{F_k} - 1\right) = \frac{F_{k+1}}{F_k} = \frac{F_{k2^0+1}}{F_{k2^0}}.$$

Next, assume that the result holds for some specific $n \geq 0$. Then for this $n$ we have

$$\Phi_n = \frac{1 + a_n}{2} = \frac{F_{k2^n+1}}{F_{k2^n}}$$

which gives

$$\begin{aligned} a_n &= \frac{2F_{k2^n+1}}{F_{k2^n}} - 1 = \frac{2F_{k2^n+1} - F_{k2^n}}{F_{k2^n}} \\ &= \frac{F_{k2^n+1} + \left(F_{k2^n+1} - F_{k2^n}\right)}{F_{k2^n}} = \frac{F_{k2^n+1} + F_{k2^n-1}}{F_{k2^n}} \end{aligned}$$

and

$$\begin{aligned} a_{n+1} &= \frac{a_n + 5/a_n}{2} = \frac{a_n^2 + 5}{2a_n} \\ &= \frac{\left(F_{k2^n+1} + F_{k2^n-1}\right)^2 + 5\left(F_{k2^n}\right)^2}{2F_{k2^n}\left(F_{k2^n+1} + F_{k2^n-1}\right)}. \end{aligned}$$

Applying the identities in Equations (6) and (7) with $m = k2^n$, we have

$$\begin{aligned} \Phi_{n+1} &= \frac{1 + a_{n+1}}{2} = \frac{1}{2} + \frac{\left(F_{k2^n+1} + F_{k2^n-1}\right)^2 + 5\left(F_{k2^n}\right)^2}{4F_{k2^n}\left(F_{k2^n+1} + F_{k2^n-1}\right)} \\ &= \frac{2F_{k2^n}\left(F_{k2^n+1} + F_{k2^n-1}\right)}{4F_{k2^n}\left(F_{k2^n+1} + F_{k2^n-1}\right)} + \frac{\left(F_{k2^n+1} + F_{k2^n-1}\right)^2 + 5\left(F_{k2^n}\right)^2}{4F_{k2^n}\left(F_{k2^n+1} + F_{k2^n-1}\right)} \\ &= \frac{\left(\left(F_{k2^n+1} + F_{k2^n-1}\right) + F_{k2^n}\right)^2 + 4\left(F_{k2^n}\right)^2}{4F_{2\cdot k2^n}} = \frac{\left(2F_{k2^n+1}\right)^2 + 4\left(F_{k2^n}\right)^2}{4F_{k2^{n+1}}} \\ &= \frac{\left(F_{k2^n+1}\right)^2 + \left(F_{k2^n}\right)^2}{F_{k2^{n+1}}} = \frac{F_{k2^{n+1}+1}}{F_{k2^{n+1}}}. \end{aligned}$$

By induction, the result holds for all $n \geq 0$ if either Condition (i) or (ii) is satisfied.

On the other hand, suppose that for some integer $k \geq 1$ we have $\Phi_n = F_{k2^n+1}/F_{k2^n}$ for all $n \geq 0$ when using a reduced fraction $a_0 = c/d$. Then for $n = 0$ we have

$$\frac{F_{k+1}}{F_k} = \Phi_0 = \frac{1+a_0}{2} = \frac{c+d}{2\,d}. \tag{8}$$

Now we simply consider all cases for the parity of $c$ and $d$. If $c$ and $d$ are both odd, then $c + d$ is even and we can simplify the fraction in (8) to

$$\frac{F_{k+1}}{F_k} = \frac{(c+d)/2}{d}. \tag{9}$$

Suppose now that $j$ divides both $d$ and $(c + d)/2$. Then $j$ will also divide $2(c + d)/2 - d = c$. Because $\gcd(c, d) = 1$, we have that $j = 1$. So both sides of Equation (9) are reduced fractions; hence, $d = F_k$, an odd Fibonacci number, and $(c + d)/2 = F_{k+1}$, which gives $c = 2F_{k+1} - d$. So Condition (i) must hold.

If one of $c$ or $d$ is even and the other is odd, then $c+d$ is odd and we again have Equation (8). But suppose $j$ divides both $c + d$ and $2\,d$. Then $j$ must be odd because $c + d$ is odd. Hence, $j$ must divide $d$. But then $j$ will divide $(c+d)-d = c$; so again $j = 1$ and $(c+d)/2d$ is completely reduced. Thus, $F_k = 2d$ is an even Fibonacci number, $d = F_k/2$, and $c = F_{k+1} - d$, which is Condition (ii), and which completes the proof.

With our original Proposition, we have Condition (ii) with $k = 3$ where $d = F_3/2 = 1$, and $c = F_4 - d = 2$. For $a_0 = 7/3$, we have Condition (i) with $d = F_4 = 3$ and $c = 2F_5 - d = 7$. We also could use something incredulous like $a_0 = 64079/28657$ to obtain $\Phi_n = F_{23\cdot 2^n+1}/F_{23\cdot 2^n}$ for all $n \geq 0$.

We now see why the pattern fails, at least for $\Phi_0$, when using $a_0 = 8/3$, $11/4$, or $12/5$ as neither Condition (i) nor Condition (ii) of the Theorem is satisfied. But with $a_0 = 5/2$ and $a_0 = 15/7$, we do establish a pattern for $n \geq 1$. For it is always the case that $\Phi_1 = (c^2 + 2cd + 5d^2)/(4cd)$. With $c = 15$ and $d = 7$, $\Phi_1$ reduces to $34/21 = F_{4\cdot 2^1+1}/F_{4\cdot 2^1}$. By the inductive argument in the proof of our theorem, a pattern holds for $n \geq 1$. However, a characterization of $c$ and $d$ that initiates the pattern for $n \geq 1$ is left as an open problem.

A similar iterative approach of approximating $\Phi$ was discovered in 1999 by J. W. Roche. When using Newton's Method of approximating the positive root of the function $f(x) = x^2 - x - 1$ with initial seed $x_1 = 2 = F_{2^1+1}/F_{2^1}$, all successive approximations are ratios of Fibonacci numbers of the form $x_n = F_{2^n+1}/F_{2^n}$. A quick proof by induction can be found in [3].

## REFERENCES


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