# COMBINATIONS WITH SUCCESSIONS AND FIBONACCI NUMBERS

**Augustine O. Munagi**
The John Knopfmacher Centre for Applicable Analysis and Number Theory,
University of the Witwatersrand Johannesburg 2050, South Africa
e-mail: munagi@maths.wits.ac.za

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## ABSTRACT

In his book *Combinatorial Identities* John Riordan considers the enumeration of $k$-combinations of $\{1,2,\ldots,n\}$ which contain a specified number of pairs of consecutive integers, or successions. We study variations and generalizations of the original idea of combinations with successions and obtain enumerative recurrences and formulas. We show that formulas for combinations with successions also enumerate a class of restricted compositions and the multi-step Fibonacci numbers.

## 1. INTRODUCTION AND PRELIMINARIES

A $k$-combination of $[n]=\{1,2,\ldots,n\}$ is any subset of $[n]$ with cardinality $k$, $0 \leq k \leq n$. It is elementary knowledge that the number of $k$-combinations of $[n]$ is the binomial coefficient $\binom{n}{k}=\frac{n.(n-1)\ldots(n-k+1)}{k!}$ which satisfies the Pascal triangle relation

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}, \qquad \binom{n}{0}=\binom{1}{1}=1, \quad \binom{0}{k}=\delta_{0k}, \tag{1}$$

where $\delta_{nm}$ is the Kronecker delta ($\delta_{nn}=1,\ \delta_{nm}=0,\ n\neq m$).

We will adopt the standard convention that the elements of every combination of $[n]$ are listed in increasing order, and $|S|$ is the cardinality of the set $S$.

A combination of $[n]$ is said to have $r$ successions ($r \geq 0$) if it contains $r$ pairs of consecutive integers, where a sequence of $u$ consecutive integers ($u \geq 1$) is considered to contain $u-1$ successions. This fundamental definition is due to Riordan [4, p.11]. The set of $k$-combinations of $[n]$ with $r$ successions will be denoted by $F_r(n,k)$, and the cardinality by $f_r(n,k)$.

**Example**: Members of $F_2(9,5)$ include $(1,2,3,5,9)$ and $(2,4,5,7,8)$.

**Theorem 1.1 ([4])**:

$$f_r(n,k)=\binom{k-1}{r}\binom{n-k+1}{k-r} \tag{2}$$

The last formula is derived in [4] as a consequence of a generating function recurrence which is equivalent to the simple assertion:

$$f_r(n,k)=f_r(n-1,k)+f_r(n-2,k-1)+f_{r-1}(n-1,k-1)-f_{r-1}(n-2,k-1),$$

$$f_r(n,1)=0(r>0), f_0(n,k)=\binom{n-k+1}{k}. \tag{3}$$

Note that $f_0(n,k)$ is the number of nonconsecutive $k$-combinations of $[n]$.

To obtain (3) one counts the $k$-combinations of $[n]$ in which $n$ is part of a succession and those in which it is not.

If $n$ is not part of a succession, then either $n$ is not selected, or $n$ is selected but $n-1$ is not selected. Thus the total number of combinations is

$$\begin{aligned}|\{b \in F_r(n,k) : n \notin b\}| + |\{b \in F_r(n,k) : n \in b \text{ and } n-1 \notin b\}| \\ = f_r(n-1,k) + f_r(n-2,k-1).\end{aligned}$$

On the other hand, the number of combinations in which $n$ is part of a succession is given by

$$\begin{aligned}|\{b \in F_r(n,k) : n, n-1 \in b\}| &= |\{b \in F_{r-1}(n-1,k-1) : n-1 \in b\}| \\ &= |F_{r-1}(n-1,k-1)| - |\{b \in F_{r-1}(n-1,k-1) : n-1 \notin b\}| \\ &= f_{r-1}(n-1,k-1) - f_{r-1}(n-2,k-1)\end{aligned}$$

$f_0(n,k)$ is completely determined by the first case, that is,

$$f_0(n,k) = f_0(n-1,k) + f_0(n-2,k-1),\ f_0(1,1) = 1,\ f_0(2,1) = 2,$$

and is easily seen to have the stated solution. Hence (3) follows.

Formula (2) may be established by showing that it satisfies (3).

Indeed we have $f_0(1,1) = \binom{0}{0}\binom{1}{1} = 1,\ f_1(1,1) = \binom{0}{1}\binom{1}{0} = 0$; thus (2) holds for $n = 1$.

Assume that (2) holds for all positive integers up to $n$. Then (3) gives

$$\begin{aligned}f_r(n+1,k) &= f_r(n,k) + f_r(n-1,k-1) + f_{r-1}(n,k-1) - f_{r-1}(n-1,k-1) \\ &= \binom{k-1}{r}\binom{n-k+1}{k-r} + \binom{k-2}{r}\binom{n-k+1}{k-r-1} + \binom{k-2}{r-1}\binom{n-k+2}{k-r} \\ &\quad - \binom{k-2}{r-1}\binom{n-k+1}{k-r} \\ &= \binom{k-1}{r}\binom{n-k+1}{k-r} + \binom{k-2}{r}\binom{n-k+1}{k-r-1} + \binom{k-2}{r-1}\binom{n-k+1}{k-r-1} \\ &= \binom{k-1}{r}\binom{n-k+1}{k-r} + \binom{k-1}{r}\binom{n-k+1}{k-r-1} \\ &= \binom{k-1}{r}\binom{n-k+2}{k-r},\end{aligned}$$

where the last three equalities have used (1). Thus (2) also holds for $n+1$, and is therefore established by mathematical induction. □

**Remark 1.2**: The following alternative recurrence is obtained by noting that $n$ is the greatest member of $v$ consecutive integers in each element of $F_r(n,k)$, where $0 \le v \le r+1$:

$$f_r(n,k) = f_r(n-1,k) + \sum_{j=0}^{r} f_{r-j}(n-j-2, k-j-1) \tag{4}$$

To see that (2) also satisfies (4) we have:

$$\begin{aligned} f_r(n-1,k) &+ \sum_{j=0}^{r} f_{r-j}(n-j-2,k-j-1) \\ &= \binom{k-1}{r}\binom{n-k}{k-r} + \sum_{j=0}^{r}\binom{k-j-2}{r-j}\binom{n-k}{k-r-1} \\ &= \binom{k-1}{r}\binom{n-k}{k-r} + \binom{k-1}{r}\binom{n-k}{k-r-1} \\ &= \binom{k-1}{r}\left(\binom{n-k}{k-r} + \binom{n-k}{k-r-1}\right) = \binom{k-1}{r}\binom{n-k+1}{k-r} = f_r(n,k), \end{aligned}$$

where the second equality follows from the identity $\binom{n}{m} = \sum_{j\ge 0}\binom{n-1-j}{m-j}$ [4, p.7].

It turns out that (4) generalizes easily to non-pairwise successions while (3) does not.

We define a distinguished subset of $F_r(n,k)$.

A combination of $[n]$ is said to have $r$ detached successions if it contains only sequences of $u$ consecutive integers, where $u = 1$ or $u = 2$. Denote the set of $k$-combinations of $[n]$ with $r$ detached successions by $Q_r(n,k)$ and let $|Q_r(n,k)| = q_r(n,k)$.

This definition shows that $q_r(n,k) = f_r(n,k)$ for $r = 0,1$. For example two elements of $Q_2(9,5)$ are $(1,2,4,6,7)$, $(2,3,5,6,9)$ while $(1,4,6,7,8) \in F_2(9,5) - Q_2(9,5)$.

**Theorem 1.3**: (i) $q_r(n,k)$ satisfies the following recurrence:

$$\begin{gathered} q_r(n,k) = q_r(n-1,k) + q_r(n-2,k-1) + q_{r-1}(n-3,k-2), \\ q_r(n,1) = 0 (r>0), q_0(n,k) = \binom{n-k+1}{k} \end{gathered} \tag{5}$$

(ii) The solution of (5) is given by

$$q_r(n,k) = \binom{k-r}{r}\binom{n-k+1}{k-r}. \tag{6}$$

The proofs of (5) and (6) are similar to those of (2) and (3). See Theorem 2.1 also. □

Observe that (2) and (6) imply

$$q_r(n,k) = \frac{(k-r)_r}{(k-1)_r} f_r(n,k),$$

where $(n)_r$ is the falling factorial defined by $(n)_r = n(n-1)\dots(n-r+1)$, $(n)_0 = 1$.

**Theorem 1.4**: The total number of combinations of $[n]$ containing detached successions is given by $\sum_k \sum_r q_r(n,k) = T_{n+2} (n \geq 0)$, where $T_n$ is the $n^{th}$ tribonacci number, defined by $T_1 = 1,\ T_2 = 1,\ T_3 = 2,\ T_n = T_{n-1} + T_{n-2} + T_{n-3}$.

**Proof**: Let $q(n) = \sum_k \sum_r q_r(n,k)$. The special values $q(0) = 1, q(1) = 2, q(2) = 4$ and $q(3) = 7$ count elements of the following sets of combinations respectively: $\{\phi\}; \{\phi, (1)\}, \{\phi, (1), (2), (1,2\}$, and $\{\phi, (1), (2), (3), (1,2), (1,3), (2,3)\}$.

It will suffice to prove that $q(n) = q(n-1) + q(n-2) + q(n-3), n \geq 3$:

$$
\begin{aligned}
&q(n-1) + q(n-2) + q(n-3) \\
&= \sum_{k\geq 0}\sum_{r\geq 0}\binom{k-r}{r}\binom{n-k}{k-r} + \sum_{k\geq 0}\sum_{r\geq 0}\binom{k-r}{r}\binom{n-k-1}{k-r} + \sum_{k\geq 0}\sum_{r\geq 0}\binom{k-r}{r}\binom{n-k-2}{k-r} \\
&= \sum_{k\geq 0}\sum_{r\geq 0}\binom{k-r}{r}\binom{n-k}{k-r} + \sum_{k\geq 1}\sum_{r\geq 0}\binom{k-r-1}{r}\binom{n-k}{k-r-1} \\
&\quad + \sum_{k\geq 2}\sum_{r\geq 1}\binom{k-r-1}{r-1}\binom{n-k}{k-r-1} \\
&= \sum_{k\geq 2}\sum_{r\geq 1}\left[\binom{k-r}{r}\binom{n-k}{k-r} + \binom{k-r-1}{r}\binom{n-k}{k-r-1} + \binom{k-r-1}{r-1}\binom{n-k}{k-r-1}\right] \\
&\quad + \sum_{k\geq 0}\left[\binom{n-k}{k} + \binom{n-k}{k-1}\right] + \binom{n}{0} + \binom{n-1}{1} + \binom{n-1}{0} \\
&= \sum_{k\geq 2}\sum_{r\geq 1}\binom{k-r}{r}\binom{n-k+1}{k-r} + \sum_{k\geq 0}\binom{n-k+1}{k} + \binom{n}{0} + \binom{n}{1} = q(n). \quad \square
\end{aligned}
$$

**Remark 1.5**: (i) Theorem 1.4 provides a non-alternating double-sum formula for the tribonacci numbers: $T_n = \sum_{k\geq 0}\sum_{r\geq 0}\binom{k-r}{r}\binom{n-k-1}{k-r}, n \geq 2$.

(ii) It is known that the tribonacci numbers $T_n$ coincide with the numbers $c(n,(1,2,3))$ of compositions of $n$ with parts in $\{1,2,3\}$ (see for example [2]). Therefore we have

$$\sum_{n\geq 0} T_{n+1}x^n = \sum_{n\geq 0} c(n,(1,2,3))x^n = \frac{1}{1-x-x^2-x^3}.$$

It follows that the generating function for $q(n) = \sum_k \sum_r q_r(n,k)$ is given by:

$$\sum_{n\geq 0} q(n)x^n = 1 + 2x + 4x^2 + \cdots = \frac{1+x+x^2}{1-x-x^2-x^3} = \frac{1-x^3}{1-2x+x^4}.$$

**Corollary 1.6**:

(i) The number of combinations of $[n]$ containing at least one succession is given by $2^n - F_{n+2}$, where $F_n$ is the $n^{\text{th}}$ Fibonacci number $(n \geq 2)$.

(ii) The number of combinations of $[n]$ containing at least one detached succession is given by $T_{n+2} - F_{n+2} (n \geq 2)$.

**Proof**: (i) From (2) we have $\sum_{k\geq 0}\sum_{r\geq 1} f_r(n,k) = \sum_{k\geq 0}\binom{n}{k} - \sum_{k\geq 0} f_0(n,k) = 2^n - F_{n+2}$, by the well-known identity (see for example [6, p.26]): $\sum_{k\geq 0} f_0(n,k) = F_{n+2}$.

(ii) As in part (i), this follows from (6) and Theorem 1.4. □

**Example**: (i) $2^4 - F_6 = 16 - 8 = 8$ counts the following combinations of $(1,2,3,4)$:

$$(1,2),(2,3),(3,4),(1,2,3),(1,2,4),(1,3,4),(2,3,4),(1,2,3,4).$$

(ii) $T_6 - F_6 = 13 - 8 = 5$ counts the following combinations of $(1,2,3,4)$:

$$(1,2),(2,3),(3,4),(1,2,4),(1,3,4).$$

The rest of this paper is organized as follows. In Section 2 we obtain generalizations of the recurrences given above, as well as a generalization of the formula for $q_r(n,k)$. Section 3 contains results which connect combinations with successions with a class of restricted compositions and the $n$-step Fibonacci numbers. Lastly, Section 4 discusses a method of characterizing $k$-combinations using partitions of the integer $k$.

The exposition mostly resembles that of a fundamental paper [3] by the author which deals with set partitions.

## 2. GENERALIZATIONS AND FURTHER RELATIONS

For any positive integers $x$ and $t$, we define a $t$-succession as the $t$ numbers $x, x+1, \ldots, x+t-1$. The combinations of $[n]$ may be classified according to the number of $t$-successions $(t \geq 1)$ appearing in a combination.

Denote the set of $k$-combinations of $[n]$ with exactly $r$ $t$-successions by $F_t(n,k,r), r \geq 0$, and let the cardinality be $f_t(n,k,r)$. Thus $f_2(n,k,r) = f_r(n,k)$.

We define two important subsets of $F_t(n,k,r)$.

For any $B \in F_t(n,k,r)$, the $r$ $t$-successions in $B$ shall be called detached if $B$ contains only $u$-successions, where $u = 1$ or $u = t$. The set of $k$-combinations of $[n]$ containing $r$ detached $t$-successions will be denoted by $Q_t(n,k,r)$ and we let $|Q_t(n,k,r)| = q_t(n,k,r)$. It follows that $q_2(n,k,r) = q_r(n,k)$.

For any $B \in F_t(n,k,r)$, the $r$ $t$-successions in $B$ are called weakly-detached if $B$ contains only $u$-successions, where $u \in [t]$. The set of $k$-combinations of $[n]$ containing $r$ weakly-detached $t$-successions is denoted by $E_t(n,k,r)$ and we let $|E_t(n,k,r)| = e_t(n,k,r)$. It follows that $Q_t(n,k,r) \subseteq E_t(n,k,r) \subseteq F_t(n,k,r)$, with $E_2(n,k,r) = Q_2(n,k,r)$, and $E_t(n,k,r) = C_t(n,k,r)$ for $(t,r) = (1,r),(t,1),(t,0)$.

For example, two elements of $F_3(14,8,2)$ are shown after each containing subset below:

$$Q_3(14,8,2) : (1,2,3,5,6,7,10,14),(2,4,5,6,9,11,12,13);$$
$$E_3(14,8,2) - Q_3(14,8,2) : (1,2,3,6,7,10,11,12),(1,2,5,6,7,9,10,11);$$
$$F_3(14,8,2) - E_3(14,8,2) : (1,2,3,4,6,8,10,12),(1,2,5,6,7,8,11,14).$$

The proofs of the following theorem are omitted since they are analogous to those of (3) and (2). But note that the derivation of (7) is contained in the proof of Theorem 2.3.

**Theorem 2.1**:
(i) $q_t(n,k,r)$ satisfies the following recurrence:

$$q_t(n,k,r) = q_t(n-1,k,r) + q_t(n-2,k-1,r) + q_t(n-t-1,k-t,r-1), \quad 1 \le r \le \lfloor k/t \rfloor,$$
$$q_t(n,k,0) = q_0(n,k) = \binom{n-k+1}{k}, q_1(n,k,r) = q_0(n,k)\delta_{kr}, q_t(n,t,r) = (n-t+1)\delta_{r1}. \tag{7}$$

(ii) The solution of (7) is given by

$$q_t(n,k,r) = \binom{k-(t-1)r}{r}\binom{n-k+1}{k-(t-1)r}. \tag{8}$$

The following corollary is immediate from (8).

**Corollary 2.2**: $q_t(n,k,r) = q_{t+j}(n+jr, k+jr, r), j = 0, \pm 1, \pm 2, \ldots$ provided the expressions are defined.

Theorem 2.1(i) is a special case of the following result (use $1 \le j \le 2$, and $e_t() \longrightarrow q_t()$).

**Theorem 2.3**: If $2 \le t \le k \le n,\ 0 \le r \le \lfloor k/t \rfloor$, then

$$e_t(n,k,r) = \sum_{j=1}^{t} e_t(n-j, k-j+1, r) + e_t(n-t-1, k-t, r-1),$$
$$e_1(n,k,r) = q_0(n,k)\delta_{kr}, e_t(n,k,0) = \sum_{j=0}^{[n/(t-1)]} e_{t-1}(m,k,j).$$

**Proof**: There are three ways to find a $b \in E_t(n,k,r)$:
(i) If $n$ is not part of a $t$-succession, then either $n$ is not selected at all or $n$ is part of a $v$-succession, where $1 \le v \le t-1$:
(a) the number of combinations without $n$ is $|\{b \in E_t(n,k,r) | n \notin b\}| = e_t(n-1,k,r)$;
(b) the number of combinations in which $n$ is part of a $v$-succession, $v < t$, is given by

$$\left| \bigcup_{j=0}^{t-2} \{B \in E_t(n,k,r) : n, n-1, \ldots, n-j \in B \text{ and } n-j-1 \in B\} \right| = \sum_{j=2}^{t} e_t(n-j, k-j+1, r).$$

(ii) The number of combinations in which $n$ is part of a (detached) succession is given by

$$|\{b \in E_t(n,k,r) : n, n-1, \ldots, n-t+1 \notin b \text{ and } n-t \in b\}| = e_t(n-t-1, k-t, r-1).$$

The main result follows by adding the combinations obtained in (i)(a), (i)(b) and (ii):

$$e_t(n,k,r) = e_t(n-1,k,r) + \sum_{j=2}^{t} e_t(n-j, k-j+1, r) + e_t(n-t-1, k-t, r-1).$$

It is clear that $e_t(n,k,0) = \sum_{j=0}^{[n/(t-1)]} e_{t-1}(m,k,j)$. The starting values are clear. □

**Theorem 2.4**: $f_3(n,k,r)$ satisfies the following recurrence:

$$f_3(n,k,r) = f_3(n-1,k,r) + f_3(n-2,k-1,r) + \sum_{j=0}^{r} f_3(n-j-3,k-j-2,r-j), \quad 1 \le r \le k-2,$$

$$f_3(n,k,0) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-j}{j}\binom{n-k+1}{k-j}.$$

**Proof**: Elements of $F_3(n,k,r)$ can be grouped into those that contain $n$ and those that do not contain $n$.
(i) Elements of $F_3(n,k,r)$ that do not contain $n$ are enumerated by $f_3(n-1,k,r)$.
(ii) If $B \in F_3(n,k,r)$ contains $n$ then $B$ contains $n$ as the greatest member of a $v$-succession, where $1 \le v \le r+2$. Then we need to count elements of the set $\{B \in F_3(n,k,r) | n \in B\}$ which is equinumerous with the disjoint union

$$\bigcup_{j=0}^{r+1} \{B \in F_3(n,k,r) : n, n-1, \ldots, n-j \in B \text{ and } n-j-1 \notin B\}.$$

The required number of elements is $f_3(n-2,k-1,r) + f_3(n-3,k-2,r) + f(n-4,k-3,r-1) + f(n-5,k-4,r-2) + \cdots + f(n-r-3,k-r-2,0)$. Note that the drop in the number $r$ of 3-successions begins with enumeration of elements of $\{B \in F_3(n,k,r) | n, n-1, n-2 \in B, n-3 \notin B\}$.

The result follows by adding the combinations obtained in (i) and (ii):

$$f_3(n,k,r) = f_3(n-1,k,r) + f_3(n-2,k-1,r) + f_3(n-3,k-2,r) + \sum_{j=4}^{r+3} f(n-j,k-j+1,r-j+3).$$

Since the absence of a 3-succession implies the presence of successions of lengths 0, 1, or 2, we have $f_3(n,k,0) = \sum_{j=0}^{\lfloor k/2 \rfloor} q_j(n,k) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k-j}{j}\binom{n-k+1}{k-j}$, where the last equality follows from (6). □

The next result can be proved by a straightforward extension of the proof of Theorem 2.4.

**Theorem 2.5**: If $1 \le t \le k \le n,\ 0 \le r \le k-t+1$, then $f_t(n,k,r)$ satisfies the relation

$$f_r(n,k,r) = \sum_{j=0}^{t-2} f_t(n-j-1,k-j,r) + \sum_{j=0}^{r} f_t(n-j-t,k-j-t+1,r-j),$$

$$f_t(n,k,0) = \sum_{j=0}^{\lfloor k/(t-1) \rfloor} e_{t-1}(n,k,j).$$

**Remark**: It would be interesting to find concise formulas for $e_t(n,k,r)$ and $f_t(n,k,r),\ t > 2$.

## 3. CONNECTIONS WITH COMPOSITIONS AND FIBONACCI NUMBERS

In this section we extend Theorem 1.4 to include connections of combinations with successions with a class of restricted compositions, and the multi-step Fibonacci numbers [5], [2].

**Theorem 3.1**: (i) The total number of combinations of $[n]$ containing detached $t$-successions is given by $\sum_k \sum_r q_t(n,k,t) = c(n+1,(1,2,t+1))(n \geq 0)$, where $c(n,(1,2,x))$ is the number of compositions of $n$ with parts in $\{1,2,x\}$, $x \geq 3$.

**Proof**: The recurrence relation for the numbers $c(n,(1,2,t+1)$ is

$$c(0,(1,2,t+1)) = 1, c(1,(1,2,t+1)) = 1, c(2,(1,2,t+1)) = 2,$$
$$c(n,(1,2,t+1)) = c(n-1,(1,2,t+1)) + c(n-2,(1,2,t+1)) + c(n-t-1,(1,2,t+1)).$$

The rest of the proof can be obtained by replacing 2 with $t$ in the proof of Theorem 1.4 since that theorem applies to $q_r(n,k) = q_2(n,k,r)$. □

**Remark 3.2**: $\sum_{n\geq 0} c(n,(1,2,t+1))x^n = \frac{1}{1-x-x^2-x^{t+1}}$. Thus if $q_t(n) = \sum_k \sum_r q_t(n,k,t)$, then $\sum_{n\geq 0}, q_t(n)x^n = \frac{1+x+x^t}{1-x-x^2-x^{t+1}}$.

**Theorem 3.3**: The total number of combinations of $[n]$ containing weakly detached $t$-successions is given by $\sum_k \sum_r e_t(n,k,t) = F_{n+2}^{(t+1)}(n \geq 0)$, where $F_n^{(m)}$ is the $n^{\text{th}}$ Fibonacci $m$-step number defined by $F_n^{(m)} = 0$, if $n \leq 0$, $F_1^{(m)} = F_2^{(m)} = 1$, $F_3^{(m)} = 2, F_n^{(m)} = \sum_{i=1}^m F_{n-i}^{(m)}$, $n \geq 4$.

**Proof**: Let $e_t(n) = \sum_k \sum_r e_t(n,k,r)$. In view of the proof of Theorem 1.4 it suffices to prove that $e_t(n) = e_t(n-1)+e_t(n-2)+\cdots+e_t(n-t-1), n \geq t+1$. We employ the recurrence relation for $e_t(n,k,r)$ derived in Theorem 2.3.

$$\begin{aligned}
\sum_{j=1}^{t+1} e_t(n-j) &= \sum_{j=1}^{t+1}\sum_{k\geq 0}\sum_{r\geq 0} e_t(n-j,k,r) = \sum_{j=1}^{t}\sum_{k\geq 0}\sum_{r\geq 0} e_t(n-j,k,r) \\
&\quad + \sum_{k\geq 0}\sum_{r\geq 0} e_t(n-t-1,k,r) \\
&= \sum_{j=1}^{t}\sum_{k\geq j-1}\sum_{r\geq 0} e_t(n-j,k-j+1,r) + \sum_{k\geq 1}\sum_{r\geq 1} e_t(n-t-1,k-t,r-1) \\
&= \sum_{j=1}^{t}\sum_{k\geq t}\sum_{r\geq 1} e_t(n-j,k-j+1,r) + \sum_{j=1}^{t}\sum_{j-1\leq k\leq t-1}\sum_{r\geq 0} e_t(n-j,k-j+1,r) \\
&\quad + \sum_{j=1}^{t}\sum_{k\geq j-1} e_t(n-j,k-j+1,0) + \sum_{k\geq t}\sum_{r\geq 1} e_t(n-t-1,k-t,r-1) \\
&= \sum_{k\geq t}\sum_{r\geq 1}\left(\sum_{j=1}^{t} e_t(n-j,k-j+1,r) + e_t(n-t-1,k-t,r-1)\right)
\end{aligned}$$

$$+\sum_{j=1}^{t}\sum_{0\le k\le t-1}\sum_{i\ge 0}e_{t-1}(n-j,k-j+1,i)+\sum_{j=1}^{t}\sum_{k\ge 0}\sum_{i\ge 0}e_{t-1}(n-j,k-j+1,i)$$

$$=\sum_{k\ge t}\sum_{r\ge 1}e_t(n,k,r)$$

$$+\sum_{0\le k\le t-1}\sum_{i\ge 0}\left(\sum_{j=1}^{t-1}e_{t-1}(n-j,k-j+1,i)+e_{t-1}(n-t,k-t+1,i)\right)$$

$$+\sum_{k\ge 0}\sum_{i\ge 0}\left(\sum_{j=1}^{t-1}e_{t-1}(n-j,k-j+1,i)+e_{t-1}(n-t,k-t+1,i)\right)$$

$$=\sum_{k\ge t}\sum_{r\ge 1}e_t(n,k,r)+\sum_{0\le k\le t-1}\sum_{i\ge 0}e_{t-1}(n,k,i)+\sum_{k\ge 0}\sum_{i\ge 0}e_{t-1}(n,k,i)$$

which reduces to $e_t(n)$. Note that we have applied the rule

$$\sum_{0\le k\le t-1}\sum_{r\ge 0}e_t(n,k,r)=\sum_{0\le k\le t-1}e_t(n,k,0)=\sum_{0\le k\le t-1}\sum_{i\ge 0}e_{t-1}(n,k,i).\quad\square$$

**Remark 3.4**: $\sum_{n\ge 0}F_{n+1}^{(t)}x^n=\frac{1}{1-x-x^2-\cdots-x^{t+1}}$, and if $e_t(n)=\sum_k\sum_r e_t(n,k,t)$, we have

$$\sum_{n\ge 0}e_t(n)x^n=\frac{1-x^{t+1}}{1-2x+x^{t+2}}.$$

**Corollary 3.5**:

(i) The combinations of $[n]$ containing at least one $t$-succession are enumerated by $2^n-F_{n+2}^{(t)}$, where $F_n^{(t)}$ is a $t$-step Fibonacci number ($t\ge 2$):
(ii) The combinations of $[n]$ containing at least one weakly-detached $t$-succession are enumerated by $F_{n+2}^{(t+1)}-F_{n+2}^{(t)}(t\ge 2)$:

**Proof**: $\binom{n}{k}=\sum_{r\ge 0}f_t(n,k,r)=f_t(n,k,0)+\sum_{r\ge 1}f_t(n,k,r)$, or, equivalently,

$$\binom{n}{k}=\sum_{j\ge 0}e_{t-1}(n,k,j)+\sum_{r\ge 1}f_t(n,k,r);$$

If we sum the last equation over $k$ and apply Theorem 3.3 we obtain part (i).

Similarly, the relation $\sum_{r\ge 0}e_t(n,k,r)=e_t(n,k,0)+\sum_{r\ge 1}e_t(n,k,r)$ gives part (ii). $\square$

**Remark 3.6**: Corollary 3.5 is equivalent to the following composite identities:

$$2^n-F_{n+2}^{(t)}=\sum_{k\ge 0}\sum_{r\ge 1}f_t(n,k,r),\qquad F_{n+2}^{(t+1)}-F_{n+2}^{(t)}=\sum_{k\ge 0}\sum_{r\ge 1}e_t(n,k,r).$$

The special cases of $t = 2$ are given by Corollary 1.6 as follows:

$$2^n - F^{(2)}_{n+2} = \sum_{k\geq 0}\sum_{r\geq 1}\binom{k-1}{1}\binom{n-k+1}{k-r}, n \geq 2;$$

$$F^{(3)}_{n+2} - F^{(2)}_{n+2} = \sum_{k\geq 0}\sum_{r\geq 1}\binom{k-r}{r}\binom{n-k+1}{k-r}, n \geq 2.$$

## 4. THE SUCCESSION STRUCTURE OF A COMBINATION

For any (ordered) $k$-combination $B$ of $[n]$, with $k > 1$, a pair $y_i, y_{i+1} \in B$ will be called separated if $y_{i+1} - y_i \geq 2$. It follows that each $k$-combination $B$ of $[n]$ possesses a unique succession (or separation) structure, or equivalently, corresponds to a unique composition $\pi = (b_1, b_2, \ldots, b_v)$ of $k$ such that each part $b_j$ represents a $b_j$-succession, $b_j > 0$, and the largest member of $b_j$ and the least member of $b_{j+1}$ are separated for all $j, 1 \leq j \leq v-1$.

For example the succession structures of (1,2,3,6,8,9) and (2,4,5,6,8,9) are (3,1,2) and (1,3,2) respectively.

Without loss of generality any succession structure can be expressed in the standard form $(b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ such that each $b_j$ represents a $b_j$-succession and appears $r_j$ times, $0 < b_1 < b_2 < \cdots < b_x$. The last expression is then the corresponding generating partition of $k$. Conversely, we can start with a partition of $k$ expressed in the form $(b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ and use it to obtain all $k$-combinations of which it is the succession structure.

For example, some 7-combinations generated by the partition $(1^2 23)$ of 7 are (1,3,4,6,8,9,10), (2,4,5,6,9,12,13), (3,4,6,8,10,11,12) and (6,11,12,20,21,22,30).

Let $\pi = (b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ represent a partition of $k$ into $v$ parts. Since a $b_j$-succession contains $b_j - 1$ 2-successions, the total number of 2-successions in a $k$-combination of $[n]$ generated by $\pi$ is $r_1(b_1-1) + \cdots + r_x(b_x - 1) = r_1 b_1 + \cdots + r_x b_x - (r_1 + \cdots + r_x) = k - v$.

Hence we have proved the following proposition:

**Proposition 4.1**: Every partition of $k$ into $v$ parts corresponds to some $k$-combination of $[n]$ containing exactly $k - v$ 2-successions, for some $n \geq k$. □

Thus the classification of the partitions of $k$ by numbers of parts implies the classification of $k$-combinations (of $[n]$ for all $n \geq k$) by numbers of 2-successions, and vice versa.

**Example**: If $k = 4$, the partitions into 2 parts (i.e. (13) and $(2^2)$) correspond to all 4-combinations containing exactly 2 2-successions which include (1,2,3,5) and (4,5,9,10).

If $k - v = r$ in Proposition 4.1, then we have $f_r(n,k) = c(k,v) f_0(n-r, k-r) = c(k,v) f_0(n-r, v)$, where $c(k,v)$ is the number of compositions of $k$ into $v$ parts [1, p. 55].

Hence Proposition 4.1 can be refined to the following statement.

**Theorem 4.2**: Every composition of $k$ into $v$ parts corresponds to some combination of $[n]$ containing exactly $k - v$ 2-successions. Moreover, a partition $\pi$ of $k$ into $v$ parts generates exactly $c(\pi)\binom{n-k+1}{v}$ $k$-combinations of $[n]$, where $c(\pi)$ is the number of permutations of the sequence of parts of $\pi$, for fixed $n (n \geq k)$.

The number $f_r(n,k)$ has a nice dual in the enumerator of ordered $k$-subsets of $[n]$ with a specified number of separations or "holes".

An ordered $k$-subset of $[n]$ has $x$ separations if it contains $x$ pairs $v_i, v_{i+1}$ such that $v_{i+1} - v_i \geq 2$. Denote the set of ordered $k$-subsets of $[n]$ with $x$ separations by $H_x(n,k)$ and let $|H_x(n,k)| = h_x(n,k)$.

For instance two elements of $H_2(11,6)$ are $\{1,2,5,9,10,11\}$ and $\{3,4,6,7,9,10\}$.

Then it is easily deduced, using Proposition 4.1, that a $k$-subset of $[n]$ with $x$ separations contains exactly $k-x-1$ 2-successions. It follows that $h_x(n,k) = f_{k-x-1}(n,k)$.

Hence $h_x(n,k) = \binom{k-1}{x}\binom{n-k+1}{x+1}$.

**Open Questions**: The problem considered in this paper can be generalized by replacing $[n]$ with a set of positive integers $V$ having $x$ separations $x \geq 0$, as defined at the end of Section 4. Denote by $f_t((|V|,x),k,r)$ the number of $k$-combinations of $V$ containing $r$ $t$-successions. Thus $f_t((|V|,0),k,r) = f_t(|V|,k,r)$.

We are unable to extend the machinery developed in this paper to handle the number $f_t((|V|,x),k,r)$ when $x > 0, t \geq 2$.

## REFERENCES


[1] G. E. Andrews. *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998.

[2] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*, Version of November 23, 2004, http://algo.inria.fr/flajolet/Publications/books.html.

[3] A. O. Munagi. "Set Partitions with Successions and Separations." *Int. J. Math. Sci.* **2005.3** (2005): 451-463.

[4] J. Riordan. *Combinatorial Identities*, Wiley, New York, 1968.

[5] E. W. Weisstein. "Fibonacci n-Step Number." From MathWorld–A Wolfram Web Resource. <http://mathworld.wolfram.com/Fibonaccin-StepNumber.html>.

[6] H. Wilf. *Generatingfunctionology*, Academic Press Inc, 1994.


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