Write $N = p_1^{2\beta_1} \cdots q_k^{2\beta_k}$, where $p, q_1, \ldots, q_k$ are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod 4$. An odd perfect number, if it exists, must have this form. McDaniel proved in 1970 that $N$ is not perfect if all $\beta_i$ are congruent to 1 (mod 3). Hagis and McDaniel proved in 1975 that $N$ is not perfect if all $\beta_i$ are congruent to 17 (mod 35). We prove that $N$ is not perfect if all $\beta_i$ are congruent to 32 (mod 65). We also show that $N$ is not perfect if all $\beta_i$ are congruent to 2 (mod 5) and either $7 | N$ or $3 | N$. This is related to a result of Iannucci and Sorli, who proved in 2003 that $N$ is not perfect if each $\beta_i$ is congruent either to 2 (mod 5) or 1 (mod 3) and $3 | N$.

1. INTRODUCTION

Write

$$N = p^{2\alpha}q_1^{2\beta_1} \cdots q_k^{2\beta_k},$$

(1.1)

where $p, q_1, \ldots, q_k$ are distinct odd primes, $\alpha, \beta_1, \ldots, \beta_k \in \mathbb{N}$, and $p \equiv \alpha \equiv 1 \pmod 4$. Euler proved that an odd perfect number, if it exists, must have the form (1.1). Let $\mathcal{O}$ denote the set of odd perfect numbers. In the case $\beta_1 = \cdots = \beta_k = \beta$, Hagis and McDaniel [3, p. 27] conjectured that $N \not\in \mathcal{O}$. This conjecture was already proved for $\beta = 1$ in 1937 [7] and for $\beta = 2$ in 1941 [5]. More recently, the conjecture has been proved for some larger values of $\beta$, including $\beta = 3, 5, 6, 8, 11, 12, 14, 17, 18, 24$, and 62 (see [1]). We now describe some infinite classes of $\beta$ for which the conjecture is known to hold. Write

$$\gamma_i := 2\beta_i + 1, \quad 1 \leq i \leq k.$$  

(1.2)

The assertion

$$d | \gamma_i \quad \text{for all } i \Rightarrow N \not\in \mathcal{O}$$  

(1.3)

was proved for $d = 3$ by McDaniel [6] in 1970, and for $d = 35$ by Hagis and McDaniel [3] in 1975. In particular, this proves the conjecture for the infinite classes $\beta \equiv 1 \pmod 3$ and $\beta \equiv 17 \pmod {35}$.

In Theorem 2 (see Section 3), we prove (1.3) for $d = 65$, which in particular proves the conjecture for all $\beta \equiv 32 \pmod {65}$. When $d$ is a product of two primes $> 3$, the only values of $d$ for which (1.3) is known are now $d = 35, 65$. There are no prime values $d > 3$ for which (1.3) is known.
Recently, Iannucci and Sorli [4] extended the result of McDaniel [6] by proving that
\[(3|N \text{ and } \gcd(\gamma_i, 15) > 1 \text{ for all } i) \Rightarrow N \not\in \mathcal{O}.\] (1.4)
(This has an important application to bounds for the number of prime factors in odd perfect numbers.) We can prove the following related results:
\[(3|N \text{ and } 7|\gamma_i \text{ for all } i) \Rightarrow N \not\in \mathcal{O},\] (1.5)
\[(7|N \text{ and } 5|\gamma_i \text{ for all } i) \Rightarrow N \not\in \mathcal{O},\] (1.6)
\[(5|N \text{ and } 77|\gamma_i \text{ for all } i) \Rightarrow N \not\in \mathcal{O},\] (1.7)
\[(3|N \text{ and } 143|\gamma_i \text{ for all } i) \Rightarrow N \not\in \mathcal{O},\] (1.8)
\[(13|N \text{ and } 55|\gamma_i \text{ for all } i) \Rightarrow N \not\in \mathcal{O}.\] (1.9)
Of the last five assertions, we prove here only (1.6); see Theorem 1. Our proofs, like the proofs of McDaniel et. al., depend on the following result of Kanold [5]:
\[(N \in \mathcal{O} \text{ and } d|\gamma_i \text{ for all } i) \Rightarrow d^4|N.\] (1.10)

2. PRELIMINARIES

Let \(\sigma(n)\) denote the sum of the positive divisors of \(n\). Assume for the purpose of contradiction that \(N \in \mathcal{O}\), so that, as in [4, eq.(2)],
\[2N = \sigma(N) = \sigma(p^\alpha) \prod_{i=1}^{k} \sigma(q_i^{2\beta_i}).\] (2.1)
Define, for prime \(q\) and integer \(d > 1,
\[f(q) := f_d(q) = \sigma(q^{d-1}) = (q^d - 1)/(q - 1)\] (2.2)
and
\[h(q) := h_d(q) = \sigma(q^{d-1})/q^{d-1}.\] (2.3)
If \(d|\gamma_i\) for all \(i\), then for all \(i\),
\[f_d(q_i) \text{ divides } f_{\gamma_i}(q_i),\] (2.4)
so \(f_d(q_i)\) divides \(N\) by (2.1) - (2.2). Since \(\alpha\) is odd,
\[(p + 1)/2 \text{ divides } \sigma(p^\alpha),\] (2.5)
so \((p + 1)/2\) divides \(N\) by (2.1). As in [4, p. 2078], it is easily seen that for odd primes \(r > q\) and integers \(a, b, c\) with \(a > 1, c > b > 1,\)
\[h_c(q) > h_b(q) > h_a(r) \geq (r + 1)/r.\] (2.6)
Moreover, for odd prime \( u \leq p \),
\[
h_a(u)(p + 1)/p \geq h_a(p)(u + 1)/u,
\]
(2.7)
since \( h_a(x)^{-1}(x + 1)/x \) is an increasing function in \( x \) for \( x > 1 \).

Let \( S \) denote the set of prime divisors of \( N \). Suppose that \( d | \gamma_i \) for all \( i \). Then by (2.1) and (2.6),
\[
2 = \frac{\sigma(N)}{N} = \frac{\sigma(p^\alpha)}{p^\alpha} \prod_{i=1}^k h_{\gamma_i}(q_i) \geq \frac{p + 1}{p} \prod_{i=1}^k h_d(q_i) = \frac{p + 1}{p} \prod_{s \in S \atop s \neq p} h_d(s).
\]
(2.8)

Let \( T \) be any subset of \( S \) containing a prime \( u \) satisfying the condition that \( u \leq p \) if \( p \in T \).
We claim that
\[
\frac{p + 1}{p} \prod_{s \in S \atop s \neq p} h_d(s) \geq \frac{u + 1}{u} \prod_{t \in T \atop t \neq u} h_d(t).
\]
(2.9)

In the case \( p \notin T \), (2.9) follows because
\[
\prod_{s \in S \atop s \neq p} h_d(s) \geq \prod_{t \in T} h_d(t) \geq \frac{u + 1}{u} \prod_{t \in T \atop t \neq u} h_d(t);
\]
in the case \( p \in T \), (2.9) follows from (2.7).

Our objective is to find a set \( T = T(d, u) \) as above such that
\[
\frac{u + 1}{u} \prod_{t \in T \atop t \neq u} h_d(t) > 2.
\]
(2.10)

In view of (2.8) - (2.9), this will provide the desired contradiction to the assumption that \( N \in \mathcal{O} \).

3. THEOREMS AND PROOFS

We begin with a lemma. Recall that \( S \) is the set of prime divisors of \( N \).

**Lemma:** If \( N \in \mathcal{O} \) and \( 13 | \gamma_i \) for all \( i \) and \( \gcd(p + 1, 21) = 1 \), then \( 13 \in S \) and \( W \subset S \), where
\[
W = \{53, 79, 131, 157, 313, 443, 521, 547, 677, 859, 911, 937, 1093, 1171, 1223, 1249, 1301, 1327, 1483, 1613, 1847\}
\]
is the set of primes \( \equiv 1 \pmod{13} \) less than 1850.
Proof: By (1.10) with $d = 13$, we have $13 \in S$. (Bold font is used to keep track of primes confirmed to lie in $S$.)

A list of primes
\[ r_1, r_2, \ldots, r_n \tag{3.1} \]
is called a $d$-chain (or simply a chain) if $r_1 \in S$ and $r_{i+1} | f_d(r_i)$ for each $i < n$, where $f_d$ is defined in (2.2). In this proof, we take $f = f_d$ with $d = 13$. If $r_i \neq p$ for each $i < n$, then every prime in the chain (3.1) lies in $S$, by (2.4). An example of a chain is
\[ 13, 264031, (882..981), 79. \tag{3.2} \]

Here $(882..981)$ is a 64-digit prime whose center digits can be easily retrieved by factoring $f(264031)$. By hypothesis, the first and third primes in (3.2) cannot be $p$, because they are $\equiv 6 \pmod{7}$. The second and fourth primes cannot be $p$ since they are $\equiv 3 \pmod{4}$. We know $13 \in S$, so $264031 \in S$ because $264031 | f(13)$. Similarly, $(882..981) \in S$ since $(882..981) | f(264031)$. Finally, $79 | f((882..981))$, so the chain (3.2) confirms that $79 \in S$.

None of the following chains can have $p$ preceding its terminal prime $r_n$, and so each chain confirms that $r_n$ (in bold) lies in $S$:

\[ 13, 53; \]
\[ 13, 264031, (882..981), 157; \]
\[ 79, (551..681), 1249; \]
\[ 79, (551..681), 50909, 499903; \]
\[ 499903, 1483; \]
\[ 499903, 32579, (313 \text{ and } 937); \]
\[ 937, 599; \]
\[ 599, 847683, (443 \text{ and } 1613); \]
\[ 599, 45137, 6397, (677 \text{ and } 911); \]
\[ 937, (111..851), 14561, 42304159; \]
\[ 42304159, 3251; \]
\[ 42304159, (766..419), (46073), (976..861), 859; \]
\[ 3251, 131; \]
\[ 1483, (301..587), 1223; \]
\[ 1223, 920011, 2081; \]
\[ 2081, (547 \text{ and } 1171); \]
\[ 157, (281..937), 5669, 168247, (395..237), 1327; \]
\[ 859, (183..471), 2029; \]
\[ 499903, 32579, (468..021). \]

Next consider the pair of chains
\[
\begin{align*}
\{313, (240..891), 9907, 1847; \\
1249, (555..427), 1847.
\end{align*}
\]
The two chains in the pair have no common primes except the terminal prime 1847. Thus, while \( p \) might precede 1847 somewhere in one chain or the other, \( p \) cannot precede 1847 in both chains. Hence (at least) one chain in the pair does not have an occurrence of \( p \) preceding 1847, and that chain confirms that 1847 \( \in S \). We now can form the single chains

\[
\begin{align*}
1847, & \quad 521; \\
521, & \quad (317..359), \quad 1951; \\
1951, & \quad (193..027), \quad 4759, \quad 1301.
\end{align*}
\]

It remains to show that 1093 \( \in S \). This is accomplished with the following pair of chains:

\[
\begin{align*}
\{ & 2029, 65677, 18038593, 1093; \\
& (468..021), 138581, (648..279), (112..139), 1873, (110..713), (582..641), \\
& (578..461), 1093. \}
\end{align*}
\]

Theorem 1: Suppose that \( 5 | \gamma_i \) for all \( i \), and \( N \in \mathcal{O} \). Then \( \gcd(N, 21) = 1 \) and \( p \equiv 1 \pmod{12} \).

Proof: By (1.10) with \( d = 5 \), we have \( 5 \in S \).

Suppose for the purpose of contradiction that \( p \equiv 2 \pmod{3} \). Then by (2.5), \( 3 \in S \). As in (2.2), write \( f = f_d \) with \( d = 5 \). Since \( f(3) = 11^2 \), (2.4) implies that \( 11 \in S \). Since \( 5 | \gamma_i \) for all \( i \) and \( 5^4 | N \) by (1.10), then, in the notation of (2.3) with \( d = 5 \), we obtain the contradiction

\[
2 = \sigma(N)/N > h(3)h(5)h(11) > 2.05.
\]

This proves that \( p \equiv 1 \pmod{12} \).

We have seen that \( 5 \in S \). We now confirm additional primes in \( S \) by using \( d \)-chains as in the Lemma, but with \( d = 5 \) instead of \( d = 13 \). The chains

\[
\begin{align*}
5, & \quad (11 \text{ and } 71); \\
11, & \quad 3221, (195..931), \quad 41;
\end{align*}
\]

confirm that 11, 71, and 41 lie in \( S \), since neither 5 nor 3221 can equal \( p \) (as \( p \equiv 1 \pmod{12} \)). Employing many such chains, we can construct a large set \( Y \) of primes in \( S \) consisting of 5 together with most of the primes \( \equiv 1 \pmod{5} \) which are \( < 10^4 \). The set \( Y \) and the long list of chains used to construct \( Y \) may be found at [2].

Suppose that \( 7 | N \). With \( T = Y \cup \{7\} \), we arrive at the contradiction (2.10) with \( u = 61, d = 5 \). Thus \( 7 \nmid N \). The same argument shows that \( 3 \nmid N \) (alternatively, \( 3 \nmid N \) follows from (1.4)). This completes the proof of Theorem 1. \( \square \)

Theorem 2: If \( 65 | \gamma_i \) for all \( i \), then \( N \notin \mathcal{O} \).

Proof: Assume for the purpose of contradiction that \( 65 | \gamma_i \) for all \( i \) and \( N \in \mathcal{O} \). From (1.10), we know that \( 13 \in S \). Let \( Y \) be as in the proof of Theorem 1, and let \( W \) be as defined in the Lemma. In view of Theorem 1, the hypotheses of the Lemma are satisfied, and so \( Y \cup W \subset S \). With

\[
T = Y \cup W \cup \{13\},
\]

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we obtain the desired contradiction (2.10) with \( u = 61, d = 65 \). This completes the proof of Theorem 2. □

REFERENCES


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