FACTORIZATIONS OF $\sum_{j=i}^{n+i-1} F_{aj-b}$

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ABSTRACT

We present one main result, the **Factorization Theorem,** which unifies several identities that exhibit factorizations of $\sum_{j=i}^{n+i-1} F_{aj-b}$. We introduce a unified proof method based on formulae for the factorization of $F_{q-d} + F_{q+d}$. One of the factors of $\sum_{j=i}^{n+i-1} F_{aj-b}$ is a member of the second order recursive sequence whose members are $\{G_1 + G_a + G_{2a} + ...\}$ or (for a even) $\{G_{\frac{a}{2}} + G_{\frac{3a}{2}} + G_{\frac{5a}{2}} + ...\}$, with G equal L or F. It is shown that, for a even, these sequences obey the same recursions as the sequences $\{G_{na}\}$.

1. THE FACTORIZATION THEOREM

Frietag [2] proved

$$\sum_{j=0}^{j=n} F_{4j+m} = F_{2n+2} F_{2n+m}. \tag{1}$$

The goal of this note is to generalize (1) by studying factorizations of $\sum_{j=i}^{j=n+i-1} F_{aj-b}$ for arbitrary integers i, a, b, n. We also provide a unified proof approach to such identities and simplified notation. All proofs in this note are based on

Lemma 1: Suppose integers p, q, r are in arithmetic progression with common difference d. Then

$$F_p + F_r = \begin{cases} L_d F_q, & \text{if } d \text{ is even,} \\ F_d L_q, & \text{if } d \text{ is odd.} \end{cases} \tag{2}$$

Proof: Equations (15a) and (15b) in [4], Chapter 3.

Lemma 1 is applied by pairing summands whose subscripts are equi-distant from a central subscript. To improve clarity we define, for integers $a \geq 1, b$ and recursive sequence $\{G_n\}$

$$G_i^{ab} = G_{ai-b}. \tag{3}$$

Assume, for convenience, $n \geq 7$ odd and a even. Then by repeatedly applying (2) and (3), we have

$$\begin{array}{lll} F^{(a,b)}_{i+\frac{n-1}{2}} & & = L_1 F^{(a,b)}_{i+\frac{n-1}{2}} \\ F^{(a,b)}_{i+\frac{n-1}{2}-1} & +F^{(a,b)}_{i+\frac{n-1}{2}+1} & = L_a F^{(a,b)}_{i+\frac{n-1}{2}} \\ F^{(a,b)}_{i+\frac{n-1}{2}-2} & +F^{(a,b)}_{i+\frac{n-1}{2}+2} & = L_{2a} F^{(a,b)}_{i+\frac{n-1}{2}} \\ F^{(a,b)}_{i+\frac{n-1}{2}-3} & +F^{(a,b)}_{i+\frac{n-1}{2}+3} & = L_{3a} F^{(a,b)}_{i+\frac{n-1}{2}} \\ & \cdots & \\ F^{(a,b)}_{i} & +F^{(a,b)}_{i+n-1} & = L_{\frac{n-1}{2}a} F^{(a,b)}_{i+\frac{n-1}{2}}. \end{array}$$

Summing the left and right sides of the above equations proves

$$\sum_{j=i}^{j=i+n-1} F^{(a,b)}_j = F_{i+\frac{n-1}{2}} \left(L_1 + L_a + L_{2a} + ... + L_{\frac{n-1}{2}a} \right).$$

To formally state the main theorem it seems useful to define, for a second order recursive sequence $\{G_n\}$ and integer $a \geq 2$,

$$\hat{G}^{(1,a)}_i = \begin{cases} G_1, \text{ if } i = 1 \\ \hat{G}^{(1,a)}_{i-1} + G_{(i-1)a}, \text{ if } i \geq 1. \end{cases} \tag{4}$$

Additionally, if a is even, we define

$$\hat{G}^{(0,a)}_i = \begin{cases} G_{\frac{a}{2}}, \text{ if } i = 1 \\ \hat{G}^{(0,a)}_{i-1} + G_{(2i-1)\frac{a}{2}}, \text{ if } i \geq 1. \end{cases} \tag{5}$$

Comment: If $i \geq 2$ then (4) becomes $\hat{L}^{(1,a)}_i = L_1 + L_a + ... + L_{(i-1)a}$. The hat of $\hat{L}^{(1,a)}_i$ is mnemonical for a 'summation' sign since the hat looks like 'half' a summation sign rotated 90 degrees. The 0 and 1 in the superscripts correspond to the n even and odd case, respectively, in the Factorization Theorem.

The Factorization Theorem: (Throughout we assume $a \geq 1$.)
(a) If $a = 1, b = 0$ and $n \equiv 2 \pmod 4$,

$$\sum_{j=i}^{n+i-1} F_j = L_{\frac{n}{2}} F_{i+1+\frac{n}{2}}.$$

(b) If $a = 1, b = 0$ and $n \equiv 0 \pmod 4$,

$$\sum_{j=i}^{n+i-1} F_j = F_{\frac{n}{2}} L_{i+1+\frac{n}{2}}.$$

(c) If n odd and a even,

$$\sum_{j=i}^{n+i-1} F_j^{(a,b)} = \hat{L}^{(1,a)}_{\frac{n+1}{2}} F^{(a,b)}_{i+\frac{n-1}{2}}.$$

(d) If n even and $a \equiv 0 \pmod 4$,

$$\sum_{j=i}^{n+i-1} F_j^{(a,b)} = \hat{L}^{(0,a)}_{\frac{n}{2}} F^{(a,b-\frac{a}{2})}_{i+\frac{n-2}{2}}.$$

(e) If n even and $a \equiv 2 \pmod 4$,

$$\sum_{j=i}^{n+i-1} F_j^{(a,b)} = \hat{F}^{(0,a)}_{\frac{n}{2}} L^{(a,b-\frac{a}{2})}_{i+\frac{n-2}{2}}.$$

Proof: Case (c), for $n \geq 7$, has been proven above using Lemma 1. The proof for $n = 1, 3, 5$ is almost identical. Similarly the proof of each of the other 4 cases of the Factorization Theorem is almost identical and omitted. Note, that case (a) (for $i = 1$) was added for completeness since it occurs as formula (38) in the appendix of [4]. Furthermore, Lemma 1 provides an alternative proof method.

2. THE $\hat{G}$ SEQUENCES

In section 1 we showed how case (c) generalizes (1). We now explore this generalization in further detail. We begin with the following definition: A sequence $\{G_n\}$ which satisfies, for all n, the recursion $G_n = cG_{n-2} + dG_{n-1}$, is said to *have type* $\langle c, d\rangle$. For example $\{F_i\}$ has type $\langle 1, 1\rangle$ and $\{F_{2i}\}$ has type $\langle -1, 3.\rangle$ The concept of *type* is useful because of the following elementary result.

Lemma 2:
(a) If two sequences have the same type and agree on two consecutive values then they are identical.
(b) Type is preserved under fixed finite linear combinations.

The following well known result will prove useful in the sequel.

Lemma 3: For fixed integers $a \geq 1$ and b, the sequence $\{G_i^{(a,b)}\}$, with $G = F$ or L, has type $\langle (-1)^{a+1}, L_a\rangle$.

Proof: [5]. The case $b = 0$ was proven by Cheves [1]. The result also follows immediately from Lemma 1.

The next two lemmas show how the $\hat{G}$ sequences generalize the sequences $\{F_i^{(a,b)}\}, \{L_i^{(a,b)}\}$.

Lemma 4: Suppose a is even and fixed: Then the sequences $\{\hat{L}_i^{(1,a)}\}, \{\hat{L}_i^{(0,a)}\}$, $\{\hat{F}_i^{(1,a)}\}$, and $\{\hat{F}_i^{(0,a)}\}$ have type $\langle -1, L_a\rangle$.

Proof: Equations (4) and (3) state that $\hat{L}_i^{(1,a)}$ is the sum of L_1 and a linear combination of $L_j^{(a,0)}, 1 \leq j \leq i-1$. Lemma 3 states that the sequence $\{L_i^{(a,0)}\}$ has type $\langle -1, L_a \rangle$. The result now follows from Lemma 1b after paying attention to the initial L_1 term. (The details are straightforward and left to the reader.) The proofs of the other cases are almost identical and omitted.

Lemma 5: For all integer i,

(a) $\hat{L}_i^{(1,2)} = L_{2i-1}$

(b) $\hat{F}_i^{(0,2)} = F_{2i}$

(c) $\hat{L}_i^{(1,4)} = F_{4i-2}$

(d) $\hat{L}_i^{(0,4)} = F_{4i}$

Proof of (a): By (4), if $i \geq 2$, $\hat{L}_i^{(1,2)} = L_1 + L_2 + L_4 + ... + L_{2(i-1)} = L_{2i-1}$. The proof of (b) is almost identical and omitted.

Proof of (c): By Lemma 3, F_{4i-2} is of type $\langle -1, L_4 \rangle$. By Lemma 4, $\hat{L}_i^{(1,4)}$ is also of type $\langle -1, L_4 \rangle$. Lemma 5(c) now follows from Lemma 1(a) after a straightforward verification that the initial values of the two sequences, for $i = 1, 2$, are equal. The proof of (d) is almost identical and omitted.

Lemma 5(c) clarifies how part (c) of the Factorization Theorem generalizes (1). One of the factors in (1) is a Fibonacci number while the other factor is a member of a $\hat{L}$ sequence (which for $a = 4$ happens to equal a member of the Fibonacci sequence). Furthermore, according to Lemma 5, the change in this factor, from a member of $\{F_i^{(a,b)}\}$, for $a = 4$, to a member of the more general $\{\hat{L}_i^{(1,a)}\}$, for $a > 4$, is only a change in the *initial* values of the sequence; it is not a change in the underlying *type*.

One final comment on the $\hat{G}$: The $\hat{G}$ resemble similar sums that occur in connection with Zeckendorf Decompositions. In this paper $\sum_{i=1}^{i+n-1} F_i^{(a,b)}$ is the *expression to be factored*; by contrast in the Zeckendorf Decomposition literature $\sum_{i=1}^{i+n-1} F_i^{(a,b)}$ only occurs as *one factor*. For example, using (3), we can formulate the following generalization of (2.3) and (4.4) of [3] for $a \equiv 2 \pmod 4$:

$$F_n^{(a,0)} = L_{\frac{a}{2}} \sum_{j=1}^{n} F_j^{(a\,\frac{a}{2})} \tag{6}$$

(The proof of this identity is left as an exercise using Lemma 1.)

3. BEST RESULTS

We next investigate whether the factorizations in the main theorem are 'best'. Table 1 exhibits numerical computations.

a	b	n	i	$\sum_{j=i}^{n+i-1} F_i^{(6,b)}$	Factor1	Factor2
6	5	1	1	1	1	1
6	5	1	2	13	1	13
6	5	1	3	233	1	233
6	5	3	1	247	19	13
6	5	3	2	4,427	19	233
6	5	3	3	79,439	19	4,181
6	4	1	1	1	1	1
6	4	1	2	21	1	21
6	4	1	3	377	1	377
6	4	3	1	399	19	21
6	4	3	2	7,163	19	377
6	4	3	3	128,535	19	6,765
6	3	1	1	2	1	2
6	3	1	2	34	1	34
6	3	1	3	610	1	610

Table 1. Numerical examples of the main theorem

As can be seen, by looking up prime factors of the center sum column, the factorizations cannot be uniformly improved (for all i) for $b = 5, 4$, but can be improved uniformly, for $b = 3$, with an additional factor of 2. In general, there are further uniform factors when $b = \frac{a}{2}$. However, the Factorization Theorem cannot be improved uniformly for all b, i. Similar heuristic remarks show the impossibility of extending the Factorization Theorem to odd $a > 1$.

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