

# BIJECTIVE PROOFS FOR FIBONACCI IDENTITIES RELATED TO ZECKENDORF'S THEOREM

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## ABSTRACT

In *Proofs that Really Count*, Benjamin and Quinn wrote that there were no known bijective proofs for certain identities that give instances of Zeckendorf's Theorem, for example,  $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$ , where  $n \geq 4$  and where  $f_k$  is the  $k$ -th Fibonacci number (there are analogous identities for  $\ell f_n$  for every positive integer  $\ell$ ). In this paper, we provide bijective proofs for  $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$  and the seven other examples of such identities listed in *Proofs that Really Count*. We interpret  $f_k$  as the cardinality of the set  $\mathcal{F}_k$  consisting of all ordered lists of 1's and 2's whose sum is  $k$ . We then demonstrate bijections that prove the eight identities listed in *Proofs that Really Count*; for example, to prove  $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$ , we give a bijection between  $\{1, 2, 3, 4, 5\} \times \mathcal{F}_n$  and  $\mathcal{F}_{n+3} \cup \mathcal{F}_{n-1} \cup \mathcal{F}_{n-4}$ . A few possible directions for future research are also given.

## 1. INTRODUCTION

We will interpret the  $n$ -th Fibonacci number  $f_n$  as the cardinality of the set  $\mathcal{F}_n$  of all ordered lists of 1's and 2's that have sum  $n$ . Thus,  $(f_0, f_1, f_2, f_3, f_4, f_5, \dots) = (1, 1, 2, 3, 5, 8, \dots)$ . For an integer  $\ell$ , the number  $\ell f_n$  will be interpreted as the cardinality of the Cartesian product  $[\ell] \times \mathcal{F}_n$ , where  $[\ell] := \{1, 2, 3, \dots, \ell\}$ . We will use the notation  $[[a_1, a_2, \dots, a_k]]$  to denote an element of  $\mathcal{F}_n$ .

On page 15 of *Proofs that Really Count* [1], the following eight identities are given under the heading of identities in need of combinatorial proofs:

$$5f_n = f_{n+3} + f_{n-1} + f_{n-4} \quad \text{for } n \geq 4 \quad (5)$$

$$6f_n = f_{n+3} + f_{n+1} + f_{n-4} \quad \text{for } n \geq 4 \quad (6)$$

$$7f_n = f_{n+4} + f_{n-4} \quad \text{for } n \geq 4 \quad (7)$$

$$8f_n = f_{n+4} + f_n + f_{n-4} \quad \text{for } n \geq 4 \quad (8)$$

$$9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (9)$$

$$10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (10)$$

$$11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (11)$$

$$12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6} \quad \text{for } n \geq 6 \quad (12)$$

(Note that each of the above identities is easily seen to be true by induction; and also each identity is true for all integers  $n$  by extending the definition of Fibonacci numbers recursively to negative indices.)

In Section 2.1, we construct a map

$$\phi_5 : [5] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+3} \cup \mathcal{F}_{n-1} \cup \mathcal{F}_{n-4},$$

and we explain why  $\phi_5$  is bijective, thus providing a bijective proof of Equation (5). In Appendix A, we define bijections  $\phi_\ell$  for  $\ell = 6, 7, 8, 9, 10, 11, 12$  which are similar to  $\phi_5$  and which provide bijective proofs of the identities given in, respectively, Equations (6), (7), (8), (9), (10), (11), and (12) above. For  $(i, X) \in [\ell] \times \mathcal{F}_n$ , the general approach for each bijection (including  $\phi_5$ ) is to prepend a short list of 1's and 2's to  $X$  depending on  $i$ , and in a few more complicated cases the short list being prepended to  $X$  also depends on the initial elements in  $X$  (which are deleted before prepending). A Maple implementation [5] of all the bijections in this paper along with programs to check bijectivity may be found online at:

<http://www.math.rutgers.edu/~matchett/Publications/FibBijections>

In Section 3 we discuss how the identities relate to Zeckendorf's Theorem and give some possible directions for further research.

## 2. A BIJECTION FOR $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$

**The bijection  $\phi_5$ :**

For  $n \geq 4$ , we define the bijection  $\phi_5 : [5] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+3} \cup \mathcal{F}_{n-1} \cup \mathcal{F}_{n-4}$  below. For  $(i, X) \in [5] \times \mathcal{F}_n$ , we define  $\phi_5$  by cases depending on the value of  $i$ . For each case, the output is  $X$  with whatever action is described, for example “prepend  $\llbracket 1, 1, 1 \rrbracket$ ” means that  $\phi_5(i, X) = Y$ , where  $Y$  is the list starting with three 1's followed by elements of  $X$  (see below for examples). Similarly, saying “change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2, 2 \rrbracket$ ” means that  $\phi(i, X) = Y$  where  $Y$  consists of two 2's followed all the elements in  $X$  except the first element (which must have been 1).

- If  $i = 1$ , then prepend  $\llbracket 1, 1, 1 \rrbracket$  ( $\hookrightarrow \mathcal{F}_{n+3}$ )
- If  $i = 2$ , then prepend  $\llbracket 1, 2 \rrbracket$  ( $\hookrightarrow \mathcal{F}_{n+3}$ )
- If  $i = 3$ , then prepend  $\llbracket 2, 1 \rrbracket$  ( $\hookrightarrow \mathcal{F}_{n+3}$ )
- If  $i = 4$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2, 2 \rrbracket$ . ( $\hookrightarrow \mathcal{F}_{n+3}$ )
  - if  $X$  starts with  $\llbracket 2, 1 \rrbracket$ , then change the  $\llbracket 2, 1 \rrbracket$  to  $\llbracket 1, 1, 2, 2 \rrbracket$ . ( $\hookrightarrow \mathcal{F}_{n+3}$ )
  - if  $X$  starts with  $\llbracket 2, 2 \rrbracket$ , then change the  $\llbracket 2, 2 \rrbracket$  to  $\llbracket \rrbracket$  ( $\hookrightarrow \mathcal{F}_{n-4}$ )
- If  $i = 5$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket \rrbracket$ . ( $\hookrightarrow \mathcal{F}_{n-1}$ )
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1, 1, 2, 1 \rrbracket$ . ( $\hookrightarrow \mathcal{F}_{n+3}$ )

We defer the description of the bijections  $\phi_\ell$  for  $\ell = 6, 7, 8, 9, 10, 11, 12$  to Section A, since they are very similar to the bijection  $\phi_5$  described above.

**Examples for  $n = 4$ :**

$$\begin{aligned}
 \phi_5 : (1, \llbracket 1, 2, 1 \rrbracket) &\mapsto \llbracket 1, 1, 1, 1, 2, 1 \rrbracket \in \mathcal{F}_7 \\
 \phi_5 : (2, \llbracket 1, 2, 1 \rrbracket) &\mapsto \llbracket 1, 2, 1, 2, 1 \rrbracket \in \mathcal{F}_7 \\
 \phi_5 : (3, \llbracket 1, 2, 1 \rrbracket) &\mapsto \llbracket 2, 1, 1, 2, 1 \rrbracket \in \mathcal{F}_7 \\
 \phi_5 : (4, \llbracket 1, 2, 1 \rrbracket) &\mapsto \llbracket 2, 2, 2, 1 \rrbracket \in \mathcal{F}_7 \\
 \phi_5 : (4, \llbracket 2, 1, 1 \rrbracket) &\mapsto \llbracket 1, 1, 2, 2, 1 \rrbracket \in \mathcal{F}_7 \\
 \phi_5 : (4, \llbracket 2, 2 \rrbracket) &\mapsto \llbracket \rrbracket \in \mathcal{F}_0 \\
 \phi_5 : (5, \llbracket 1, 2, 1 \rrbracket) &\mapsto \llbracket 2, 1 \rrbracket \in \mathcal{F}_3 \\
 \phi_5 : (5, \llbracket 2, 1, 1 \rrbracket) &\mapsto \llbracket 1, 1, 2, 1, 1, 1 \rrbracket \in \mathcal{F}_7
 \end{aligned}$$

Note that for  $n = 4$ , the domain for  $\phi_5$  is  $\{1, 2, 3, 4, 5\} \times \mathcal{F}_4$  and the range is  $\mathcal{F}_7 \cup \mathcal{F}_3 \cup \mathcal{F}_0$ .

### 2.1.1 Showing $\phi_5$ is bijective

Each of the eight cases for  $\phi_5$  is injective. Thus, it is sufficient to check that none of the eight images overlap and that together they are surjective, which we now do.

There is exactly one case mapping to  $\mathcal{F}_{n-4}$  (namely, when  $i = 4$  and  $X$  starts with  $[[2,2]]$ ), and it is clearly surjective. Also, there is exactly one case mapping to  $\mathcal{F}_{n-1}$  (namely, when  $i = 5$  and  $X$  starts with  $[[1]]$ ), and it is also clearly surjective. There are six cases mapping to  $\mathcal{F}_{n+3}$ , and one can compare the prefixes attached in each case to see that the six images are disjoint and together comprise all of  $\mathcal{F}_{n+3}$  (see Figure 1).

$$\mathcal{F}_{n+3}$$

Figure 1: All of the elements of  $\mathcal{F}_{n+3}$  fit into the tree above. To find the placement of a particular ordered list  $X \in \mathcal{F}_{n+3}$ , move down the tree from the root,  $\circ$ , at the top, moving left in the  $k$ -th level if the  $k$ -th entry in  $X$  is 1 and moving right if the  $k$ -th entry is 2.

Thus,  $\phi_5$  is a bijection that provides a combinatorial proof of Equation (5). It is clear that a combinatorial inverse for  $\phi_5$  may be constructed using the tree in Figure 1.

## 3. FURTHER DIRECTIONS

As mentioned in [1], the identities in Equations (5), (6), ..., (12), give instances of Zeckendorf's Theorem that every positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers (for more on Zeckendorf's Theorem, see [2]; a clever combinatorial proof has also been given by Gessel [3]).

We may define the  $\ell$ -th *Zeckendorf family identity* to be an equation with the form

$$\ell f_n = \sum_{t \in S_\ell} f_{n+t} \tag{13}$$

that holds for all integers  $n$ , where  $S_\ell$  is a finite subset of integers that depends only on  $\ell$  and contains no two adjacent integers. Plugging in values for  $\ell$  and  $n$  in the  $\ell$ -th Zeckendorf family identity gives a formula that is an instance of Zeckendorf's Theorem (so long as  $n$  is sufficiently

large—Zeckendorf's Theorem only applies to Fibonacci numbers with positive indices, so we need  $n + t > 0$  for all  $t \in S_\ell$ ). However, that fact that Zeckendorf family identities exist for all  $\ell$  is not simply a consequence of Zeckendorf's Theorem, since the  $\ell$ -th Zeckendorf family identity holds for *all* sufficiently large  $n$  while the set  $S_\ell$  is fixed. Also, note that the set  $S_\ell$  is unique (by Zeckendorf's Theorem).

The bijections described in this paper may appear to involve some arbitrary choices, and indeed they do. For the bijection  $\phi_\ell$  with domain  $[\ell] \times \mathcal{F}_n$ , the elements in  $[\ell]$  can be regarded as formal symbols; and thus there are  $\ell!$  bijections possible by renaming the symbols in  $[\ell]$ , which simply permutes the main cases of the bijection  $\phi_\ell$ . Also, by reversing all lists in the range or domain or both it is trivially possible to construct new bijections; however, these new bijection do not contribute to understanding the structure of the Zeckendorf family identities. Thus, we will define two bijections for the  $\ell$ -th Zeckendorf family identity to be *formally equivalent* if one may be derived from the other by permuting the symbols in  $[\ell]$  and possibly reversing lists in the range or domain.

Interestingly, the methods used to construct the bijections in this paper can be used to construct numerous bijections that are not formally equivalent. For example, it is possible to construct (at least)  $3! = 6$  bijections for Equation (5) that are distinct with respect to formal equivalence (one of which is  $\phi_5$ ). These bijections provide similar, but distinct, bijective proofs of Equation (5). Analogously, counting only bijections that are not formally equivalent, there are (at least) 6 possible bijections for each of Equations (6), (7), and (8); (at least)  $4! = 24$  possible bijections for each of Equations (9), (10), and (11); and (at least)  $8! = 40320$  possible bijections for Equation (12).

Paul Raff [4] proposed an alternative approach that can be used to construct a bijection for Equation (5), and it turns out that the resulting bijection is formally equivalent to one of the 6 bijections constructible by the methods in the current paper.

The following questions are of interest for further work.

1. Is there another approach for constructing bijections for Equations (5), (6),  $\dots$ , and (12), or other Zeckendorf family identities that explicitly uses Zeckendorf's condition that the Fibonacci numbers not be adjacent?

Note that the bijection  $\phi_5$  (see Section 2.1) does *not* make explicit use of the fact that the sets  $\mathcal{F}_{n+3}$ ,  $\mathcal{F}_{n-1}$ , and  $\mathcal{F}_{n-4}$  have indices that differ by at least 2.

2. Is there another approach for constructing bijections for Equations (5), (6),  $\dots$ , and (12), or other Zeckendorf family identities where  $[\ell]$  is interpreted combinatorially, instead of as just a collection of formal symbols?
3. Is there a systematic way to construct a (not necessarily unique) bijection for the  $\ell$ -th Zeckendorf family identity (see Equation (13))?
4. Is there a unique or canonical bijection for the  $\ell$ -th Zeckendorf family identity?
5. Is there a simple combinatorial proof of the existence and uniqueness of the  $\ell$ -th Zeckendorf family identity?

A possible starting point might be Gessel's elegant combinatorial proof for Zeckendorf's Theorem and a few of its generalizations ([3]). The basic idea of Gessel's proof is to form two simple bijections—one from Zeckendorf representations to the ordered list  $L$  of positive binary numbers with no adjacent 1's, and another from  $L$  to the positive integers—and then to prove that the composition of the two bijections is the canonical map taking a Zeckendorf representation to the integer it represents.

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## APPENDIX

### Bijections for Equations (6), (7), (8), (9), (10), (11), and (12):

In this appendix we describe bijections  $\phi_\ell$  for  $\ell = 6, 7, 8, 9, 10, 11, 12$ , which provide bijective proofs of, respectively, Equations (6), (7), (8), (9), (10), (11), and (12). It is clear that the  $\phi_\ell$  defined below are bijections by drawing trees analogous to the one in Figure 1.

#### A.1 A bijection for $6f_n = f_{n+3} + f_{n+1} + f_{n-4}$ :

We define the bijection  $\phi_6 : [6] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+3} \cup \mathcal{F}_{n+1} \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (6). For  $(i, X) \in [6] \times \mathcal{F}_n$ , we define  $\phi_6$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+1})$
- If  $i = 2$ , then prepend  $\llbracket 1, 1, 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+3})$
- If  $i = 3$ , then prepend  $\llbracket 1, 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+3})$
- If  $i = 4$ , then prepend  $\llbracket 2, 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+3})$
- If  $i = 5$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2, 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+3})$
  - if  $X$  starts with  $\llbracket 2, 1 \rrbracket$ , then change the  $\llbracket 2, 1 \rrbracket$  to  $\llbracket 1, 1, 2, 2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+3})$
  - if  $X$  starts with  $\llbracket 2, 2 \rrbracket$ , then change the  $\llbracket 2, 2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$
- If  $i = 6$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+1})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1, 1, 2, 1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+3})$

#### A.2 A bijection for $7f_n = f_{n+4} + f_{n-4}$ :

We define the bijection  $\phi_7 : [7] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+4} \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (7). For  $(i, X) \in [7] \times \mathcal{F}_n$ , we define  $\phi_7$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1, 1, 1, 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 2$ , then prepend  $\llbracket 1, 1, 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 3$ , then prepend  $\llbracket 1, 2, 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 4$ , then prepend  $\llbracket 2, 1, 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 5$ , then prepend  $\llbracket 2, 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 6$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1, 2, 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2, 1 \rrbracket$ , then change the  $\llbracket 2, 1 \rrbracket$  to  $\llbracket 1, 1, 1, 2, 2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2, 2 \rrbracket$ , then change the  $\llbracket 2, 2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$
- If  $i = 7$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2, 1, 2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1, 1, 1, 2, 1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$

**A.3 A bijection for  $8f_n = f_{n+4} + f_n + f_{n-4}$  :**

We define the bijection  $\phi_8 : [8] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+4} \cup \mathcal{F}_n \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (8). For  $(i, X) \in [8] \times \mathcal{F}_n$ , we define  $\phi_8$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 2$ , then prepend  $\llbracket 1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 3$ , then prepend  $\llbracket 1,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 4$ , then prepend  $\llbracket 2,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 5$ , then prepend  $\llbracket 2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 6$ , then do nothing  $(\hookrightarrow \mathcal{F}_n)$
- If  $i = 7$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 1,1,1,2,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$
- If  $i = 8$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1,1,1,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$

**A.4 A bijection for  $9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4}$ :**

We define the bijection  $\phi_9 : [9] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+4} \cup \mathcal{F}_{n+1} \cup \mathcal{F}_{n-2} \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (9). For  $(i, X) \in [9] \times \mathcal{F}_n$ , we define  $\phi_9$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 2$ , then prepend  $\llbracket 1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 3$ , then prepend  $\llbracket 1,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 4$ , then prepend  $\llbracket 2,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 5$ , then prepend  $\llbracket 2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 6$ , then prepend  $\llbracket 1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+1})$
- If  $i = 7$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,1,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-2})$
- If  $i = 8$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 2,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+1})$
  - if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$
- If  $i = 9$ , then
  - if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$
  - if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 2,1,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+1})$
  - if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket 2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+1})$

**A.5 A bijection for  $10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4}$ :**

We define the bijection  $\phi_{10} : [10] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+4} \cup \mathcal{F}_{n+2} \cup \mathcal{F}_{n-2} \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (10). For  $(i, X) \in [10] \times \mathcal{F}_n$ , we define  $\phi_{10}$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 2$ , then prepend  $\llbracket 1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 3$ , then prepend  $\llbracket 1,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$
- If  $i = 4$ , then prepend  $\llbracket 2,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$

- If  $i = 5$ , then prepend  $\llbracket 2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 6$ , then prepend  $\llbracket 1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+2})$   
 If  $i = 7$ , then prepend  $\llbracket 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+2})$   
 If  $i = 8$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,1,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-2})$   
 If  $i = 9$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+2})$   
 • if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$   
 If  $i = 10$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+2})$

**A.6 A bijection for  $11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4}$ :**

We define the bijection  $\phi_{11} : [11] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+4} \cup \mathcal{F}_{n+2} \cup \mathcal{F}_n \cup \mathcal{F}_{n-2} \cup \mathcal{F}_{n-4}$  below. This bijection is a combinatorial interpretation of Equation (11). For  $(i, X) \in [11] \times \mathcal{F}_n$ , we define  $\phi_{11}$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 2$ , then prepend  $\llbracket 1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 3$ , then prepend  $\llbracket 1,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 4$ , then prepend  $\llbracket 2,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 5$ , then prepend  $\llbracket 2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 If  $i = 6$ , then prepend  $\llbracket 1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+2})$   
 If  $i = 7$ , then prepend  $\llbracket 2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+2})$   
 If  $i = 8$ , then do nothing  $(\hookrightarrow \mathcal{F}_n)$   
 If  $i = 9$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,1,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-2})$   
 If  $i = 10$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 1,2,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+2})$   
 • if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-4})$   
 If  $i = 11$ , then  
 • if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+4})$   
 • if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+2})$

**A.7 A bijection for  $12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6}$ :**

We define the bijection  $\phi_{12} : [12] \times \mathcal{F}_n \longrightarrow \mathcal{F}_{n+5} \cup \mathcal{F}_{n-1} \cup \mathcal{F}_{n-3} \cup \mathcal{F}_{n-6}$  below. This bijection is a combinatorial interpretation of Equation (12). For  $(i, X) \in [12] \times \mathcal{F}_n$ , we define  $\phi_{12}$  by cases depending on the value of  $i$ .

- If  $i = 1$ , then prepend  $\llbracket 1,1,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 2$ , then prepend  $\llbracket 1,1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 3$ , then prepend  $\llbracket 1,1,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 4$ , then prepend  $\llbracket 1,2,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 5$ , then prepend  $\llbracket 2,1,1,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 6$ , then prepend  $\llbracket 1,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$   
 If  $i = 7$ , then prepend  $\llbracket 2,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$

- If  $i = 8$ , then prepend  $\llbracket 2,2,1 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$
- If  $i = 9$ , then
- if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,1,1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$
  - if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket \ \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-3})$
  - if  $X$  starts with  $\llbracket 2,2,1 \rrbracket$ , then change the  $\llbracket 2,2,1 \rrbracket$  to  $\llbracket 2,2,2,2,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+5})$
  - if  $X$  starts with  $\llbracket 2,2,2 \rrbracket$ , then change the  $\llbracket 2,2,2 \rrbracket$  to  $\llbracket \ \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-6})$
- If  $i = 10$ , then
- if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket \ \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n-1})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 2,2,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+5})$
- If  $i = 11$ , then
- if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 2,1,1,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$
  - if  $X$  starts with  $\llbracket 2 \rrbracket$ , then change the  $\llbracket 2 \rrbracket$  to  $\llbracket 1,1,2,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+5})$
- If  $i = 12$ , then
- if  $X$  starts with  $\llbracket 1 \rrbracket$ , then change the  $\llbracket 1 \rrbracket$  to  $\llbracket 1,2,1,2 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+5})$
  - if  $X$  starts with  $\llbracket 2,1 \rrbracket$ , then change the  $\llbracket 2,1 \rrbracket$  to  $\llbracket 1,1,2,2,2 \rrbracket$   $(\hookrightarrow \mathcal{F}_{n+5})$
  - if  $X$  starts with  $\llbracket 2,2 \rrbracket$ , then change the  $\llbracket 2,2 \rrbracket$  to  $\llbracket 2,2,2,2,1 \rrbracket$ .  $(\hookrightarrow \mathcal{F}_{n+5})$

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