ASYMPTOTIC BEHAVIOR OF CERTAIN DUCCI SEQUENCES

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ABSTRACT

The Ducci map applied to vectors in $\mathbb{R}^3$ is considered. It is shown that for any starting vector, the corresponding sequence of iterates either eventually becomes periodic or, if not eventually periodic, asymptotically approaches the zero vector. A precise characterization of which vectors exhibit each of these separate behaviors is given.

1. INTRODUCTION

Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$f(a, b, c, d) = (|a - b|, |b - c|, |c - d|, |d - a|),$$

and for a fixed starting vector, form the sequence $\{f^n(a, b, c, d)\}_{n=0}^{\infty}$ in $\mathbb{R}^4$. It was aquestion about the limiting behavior of such sequences which Ciamberlini and Marengoni [16] attributed to E. Ducci in 1937. Since then, such sequences have most commonly come to be known as *Ducci sequences* or the *four-number game*.

The literature devoted to this topic has become quite extensive. This seems in part to be due to the ease with which the question can be posed, particularly in the case of vectors in $\mathbb{Z}^4$. For example, by experimenting with a variety of different vectors $(a, b, c, d) \in \mathbb{Z}^4$, one will likely find that the corresponding Ducci sequence reaches the vector $(0, 0, 0, 0)$ in a relatively small number of steps. The question of whether or not this must always occur has appeared in a number of books on mathematics meant for general audiences [28, 29, 45, 46, 51], as well as at various times in the mathematics literature [1, 24, 27, 47].

That the zero vector will indeed always be reached after a finite number of iterations of f applied to vectors in $\mathbb{Z}^4$is confirmed in [28]. And in [47], a more general question also appears. For any integer $n \geq 2$, define the map $f : \mathbb{Z}^n \to \mathbb{Z}^n$ by making the obvious generalization to get

$$f(x_1, x_2, \dots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \dots, |x_n - x_1|).$$

The question then becomes, for any starting vector $(x_1, x_2, \dots, x_n)$, does the corresponding Ducci sequence become the zero vector after only a finite number of iterations? The original proof that this is in fact the case if and only if n is a power of 2 appeared in the seminal paper of Ciamberlini and Marengoni [16], and was followed by a surprising variety of later proofs using various techniques [2, 10, 12, 20, 22, 26, 40, 53].

One way in which these results have been refined has been to consider the number of iterations needed to reach the zero vector when n is a power of 2. In [53] a bound is given for the number of such iterations. And in the case of $n = 4$, [4, 19, 46] show that for any $M \in \mathbb{N}$, there exist integer vectors which take at least M iterations to reach $(0, 0, 0, 0)$. Even more specifically, [5, 49] use the Tribonacci numbers to show how to find a starting vector in $\mathbb{Z}^4$ which takes any specified number of iterations to reach the zero vector, with [42] giving similar results. Many of these results are generalized to arbitrary powers of 2 in]10, 12, 22].

When n is not a power of 2 and integer entries are considered, the sequence of iterates of f, although not necessarily convergent to zero, still displays an eventual regularity. In particular, for any integer $n \geq 2$, if $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, then the corresponding Ducci sequence will eventually contain a vector which is a scalar multiple of a vector in $\{0, 1\}^n$ [21, 22, 23, 43]. It is clear that any such 0-1 vector will then cycle under iteration by f, and so in this sense all integer Ducci sequences have tails which consist of repeating segments. Of course, in the case that n is a power of 2 these segments contain only the zero vector. Also in analogy with the power of 2 case, for any $n \geq 2$ it has been shown that there exist initial vectors which can take any specified number of iterations to reach one of these repeating segments [65, 17, 18, 22, 33, 34]. Others have taken up questions related to these repeating segments, such as their lengths [7, 10, 11, 21, 25, 36, while yet others have generalized to consider the Ducci map over more general abelian groups [8]. Rather than restricting attention to vectors with integer entries, in this paper we will be interested in the Ducci map applied to vectors with real-number entries.

Perhaps somewhat surprisingly, the greatest amount of work on the Ducci map applied to vectors with real entries has occurred for $n = 4$. In fact, Lotan [32] proved that the situation here differs from the integer-entry case in that there exist vectors in $\mathbb{R}^4$ for which the corresponding Ducci sequence never reaches the zero vector. In particular, by letting q represent the positive solution to the cubic equation $x^3 - x^2 - x - 1 = 0$, the vector $(1, q, q^2, q^3)$ will correspond to a Ducci sequence which never reaches $(0, 0, 0, 0)$. In using the Tribonacci numbers to provide vectors in $\mathbb{Z}^4$ which take arbitrarily long to reach $(0, 0, 0, 0)$, [49] makes this same observation, and shows that the Tribonacci numbers can be used to obtain increasingly accurate approximations to q. Furthermore, [32] proved that every vector in $\mathbb{R}^4$, except ones which can be obtained from $(1, q, q^2, q^3)$ through the obvious transformations (translating, shifting, reflecting, and scaling), will reach $(0, 0, 0, 0)$ in a finite number of steps. Therefore, modulo these exceptional vectors, of which there is essentially only one, the behavior of f on $\mathbb{R}^4$ is the same as on $\mathbb{Z}^4$. But, although the vector $(1, q, q^2, q^3)$ does not reach the zero vector in a finite number of iterations, the corresponding Ducci sequence does converge to $(0, 0, 0, 0)$. Consequently, even in this exceptional case, the asymptotic behavior is consistent with the behavior of all other vectors. It is also interesting to note that this result has been, in whole or in part, proved and reproved a number of times since [3, 12, 30, 31, 37, 38, 46]. In [48] the probabilities of randomly choosing a vector in $\mathbb{R}^4$ which takes any specified number of iterations to reach the zero vector are calculated. Further mention of some of the previous work on this problem can be found in [14, 39].

A variety of generalizations of the Ducci map problem have also been considered. One such example is that of a Ducci process [52]. Here, rather than the map which takes $(x_1, x_2, \ldots, x_n)$ to $(|x_1 - x_2|, \ldots, |x_n - x_1|)$, the function $(x, y) \mapsto |x - y|$ is replaced by a more general map $\phi(x, y)$. Then the Ducci process is such that $(x_1, x_2, \ldots, x_n)$ maps to the vector $(\phi(x_1, x_2), \ldots, \phi(x_n, x_1))$. Such generalized mappings were further considered by [35, 44]. Other generalizations of the original Ducci map utilize weightings. For example, the original mapping utilizes the weighting $(1, -1)$, whereas one other possibility which has recently been considered [13, 15] utilizes the weighting $(-1, 2, -1)$ to define the map on vectors in $\mathbb{R}^3$ where $(x_1, x_2, x_3) \mapsto (|2x_1 - x_3 - x_2|, |2x_2 - x_1 - x_3|, |2x_3 - x_2 - x_1|)$.

For the present work, attention will be focused on the Ducci map f applied to vectors in $\mathbb{R}^3$. In light of the work done on the $\mathbb{R}^4$ case, it is quite surprising that very little has appeared which explicitly treats real-valued entries for $n = 3$. One exception is [50], where the question of which such sequences result in cycles is considered. Another exception is [41]. There, among other things, it was shown that if n is a power of 2, then any vector $v \in \mathbb{R}^n$ will give rise to a Ducci sequence for which $\lim\{|f^n(v)|\} = 0$. Then, in [9], this was further generalized to show that for any positive integer $n \geq 2$ and any vector $v \in \mathbb{R}^n$, the sequence $\{f^n(v)\}$ converges to a periodic vector. In the case that n is a power of 2, [41] tells us that this periodic vector is the trivial one, namely $(0, 0, \ldots, 0)$, whereas otherwise it will be a vector which is a scalar multiple of an element in $\{0, 1\}^n$.

Although [9] sheds some light on the behavior of the Ducci map for $n = 3$, it still leaves open certain questions regarding the behavior in this case. For example, [9] constructs a vector in $\mathbb{R}^7$ which corresponds to a sequence which asymptotically approaches a nontrivial periodic vector, but which has none of its entries themselves periodic. We will show that this behavior cannot occur for $n = 3$. That is, either a vector in $\mathbb{R}^3$ gives rise to a sequence which is eventually periodic, or it asymptotically approaches the trivial periodic vector $(0, 0, 0)$. Furthermore, we characterize precisely which vectors in $\mathbb{R}^3$ exhibit each of these separate behaviors.

2. DUCCI SEQUENCES IN $\mathbb{R}^3$

From this point forward we consider the Ducci map f defined on $\mathbb{R}^3$. As shown in [21, 22, 23, 43], for any vector $v \in \mathbb{Z}^3$, there exists $n \in \mathbb{N}$ such that $f^n(v)$ is a scalar multiple of a vector in $\{0, 1\}^3$. Clearly this result also holds for vectors with rational entries. In fact, given $v \in \mathbb{R}^3$,$v = (a, b, c)$, if there exist $\alpha, x \in \mathbb{R}$, $\alpha \neq 0$, such that $\alpha(v + (x, x, x)) \in \mathbb{Q}^3$, then since $f(\alpha(v + (x, x, x))) = |\alpha|(|a - b|, |b - c|, |c - a|) = |\alpha| f(v)$, we see that any such vector will also correspond to a Ducci sequence $\{f^n(v)\}$ which will eventually be periodic. Of course, there are many vectors in $\mathbb{R}^3$ which do not have this property, such as $(0, 1, \sqrt{2})$. Therefore, to give a complete analysis of the behavior of f, we are led to a consideration of this latter type of vector.

Definition 2.1: We will call a vector $v \in \mathbb{R}^3$ *heterogeneous* if for all $x, \alpha \in \mathbb{R}$, $\alpha \neq 0$, we have$\alpha(v + (x, x, x)) \notin \mathbb{Q}^3$.

The following basic result will be important for us as we continue our analysis. Since those vectors which cycle are multiples of vectors in $\{0, 1\}^3$, and are therefore not heterogeneous [50], the following result also implies that only non-heterogeneous vectors give rise to Ducci sequences which will reach a vector which is in a cycle.

Lemma 2.2: *The vector $v \in \mathbb{R}^3$ is heterogeneous if and only if $f(v)$ is heterogeneous.*

Proof: It is clear that if there exist $\alpha, x \in \mathbb{R}$,$\alpha \neq 0$, such that $\alpha(v + (x, x, x)) \in \mathbb{Q}^3$, then, as our above calculations demonstrate, $f(v)$ will not be heterogeneous.

Before proceeding with the converse, we remark that a rearrangement of the entries in a vector prior to application of f will not alter the magnitudes of the entries in the image, only their relative positions within the vector, and therefore will have no effect on the matter of whether or not $f(v)$ is heterogeneous. As such, given $v = (a, b, c) \in \mathbb{R}^3$, we may assume, without a loss of generality, that $a \le b \le c$. We further assume that $\alpha \ne 0$ and $x \in \mathbb{R}$ have been found such that

$$\alpha[f(v) + (x, x, x)] = (\alpha(b - a + x), \alpha(c - b + x), \alpha(c - a + x)) \in \mathbb{Q}^3.$$

Then, for example, $\alpha(c-a+x) - \alpha(c-b+x) = \alpha(b-a) \in \mathbb{Q}$, and in a similar fashion we can show that $\alpha(c - a) \in \mathbb{Q}$. Thus,

$$\alpha(v - (a, a, a)) = \alpha(0, b - a, c - a) \in \mathbb{Q}^3,$$

showing v is not heterogeneous. $\square$

Remark: A more general result is possible here. Specifically, one can show that if n is odd, then $v \in \mathbb{R}^n$ is heterogeneous if and only if $f(v)$ is heterogeneous. As [32] has shown, this is not true if n is a power of 2. However, the analogous result for n even but not a power of 2 is still open.

We now define two functions which will allow us to shift our analysis of the Ducci mapping f to an analysis of a mapping defined on a subset of the plane $\mathbb{R}^2$. Specifically, define g on the set of heterogeneous vectors in $\mathbb{R}^3$ which have one zero entry and two positive (necessarily distinct) entries to those points in the plane $\{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\}$ by $g((0, a, b)) = (a, b)$. Considering this subset of $\mathbb{R}^2$, define the function

$$h : \{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\} \to \{(a, b) : a/b \notin \mathbb{Q}, 0 < a < b\}$$

by

$$h(a, b) = \begin{cases} (2a - b, a), & \text{if } b < 2a; \\ (b - 2a, b - a), & \text{if } 2a < b. \end{cases}$$

The function h is the analog of the Ducci map f restricted to heterogeneous vectors, a statement we make more precise in the following.

Lemma 2.3: *Suppose v is a heterogeneous vector in $\mathbb{R}^3$. Then there exist unique $l, L \in \mathbb{R}$ such that $g(f(v) - (L, L, L)) = h(g(v - (l, l, l)))$. In particular, l equals the smallest entry of v and L the smallest entry of $f(v)$.*

Proof: Suppose $v = (a, b, c)$, where, without a loss of generality we assume $a < b < c$. As $f(v) = f(v + (j, j, j))$ for any real j, we let $j = -a$ to obtain $v + (j, j, j) = (0, b - a, c - a) = (0, \alpha, \beta)$, where $\alpha = b - a$ and $\beta = c - a$, so that $f((0, \alpha, \beta)) = f(v)$. Thus, our result will follow if we can find l and L such that $g(f((0, \alpha, \beta)) - (L, L, L)) = h(g(v - (l, l, l)))$.

Calculating, we see that $f((0, \alpha, \beta)) = (\alpha, \beta - \alpha, \beta)$. For $f((0, \alpha, \beta)) - (L, L, L)$ to be in the domain of g, one of its entries must be zero. As β is the greatest of the entries of $f((0, \alpha, \beta))$, and in fact strictly larger than α and $\beta - \alpha$ since$v$ is heterogeneous, we have only two cases to consider.

Case 1: If $2\alpha > \beta$, and therefore $\alpha > \beta - \alpha$, then set $L = \beta - \alpha$, and $f((0, \alpha, \beta)) - (L, L, L) = (2\alpha - \beta, 0, \alpha)$. Note that $\alpha > 2\alpha - \beta$ since this is equivalent to $\beta > \alpha$. Also note that since $(0, \alpha, \beta)$ is heterogeneous, so too is $f((0, \alpha, \beta))$, and thus $f((0, \alpha, \beta)) - (L, L, L)$ is in the domain of g.

Case 2: If $2\alpha < \beta$, and therefore $\alpha < \beta - \alpha$, then set $L = \alpha$, and $f((0,\alpha,\beta)) - (L,L,L) = (0, \beta - 2\alpha, \beta - \alpha)$. Clearly $\beta - \alpha > \beta - 2\alpha$. The previous comments regarding the fact that $f((0,\alpha,\beta)) - (L,L,L)$ is in the domain of g apply here as well.

Thus we have

$$g(f(v) - (L,L,L)) = \begin{cases} (2\alpha - \beta, \alpha), & \text{if } 2\alpha > \beta \\ (\beta - 2\alpha, \beta - \alpha), & \text{if} 2\alpha < \beta, \end{cases}$$

and upon substituting for α and β, we find that

$$g(f(v) - (L,L,L)) = \begin{cases} (2b - a - c, b - a), & \text{if } 2b > a + c \\ (c + a - 2b, c - b), & \text{if } 2b < a + c. \end{cases}$$

On the other hand, we have $g(v - a) = (b - a, c - a)$, and so by now calculating $h(g(v - (l,l,l)))$ for $l = a$, we reach the desired conclusion. $\square$

This lemma can be generalized to apply to any iterate of f, asstated in the following lemma. It is then on the basis of Lemma 2.4 that we are able to analyze the action of h in order to determine the behavior of f on heterogeneous vectors.

Lemma 2.4: *Let $n \in \mathbb{N}$ and suppose v is a heterogeneous vector in $\mathbb{R}^3$. Then there exist unique real numbers l, L such that*

$$g(f^n(v) - (L,L,L)) = h^n(g(v - (l,l,l))).$$

Here l is the smallest entry of v and L is the smallest entry of $f^n(v)$.

3. ASYMPTOTIC BEHAVIOR OF HETEROGENEOUS VECTORS

We now focus our analysis on those points $(a,b) \in \mathbb{R}^2$ such that $0 < a < b$ and $a/b \notin \mathbb{Q}$. To this end, the next lemma demonstrates that any such point with $b < 2a$ will eventually be shifted under iteration by h to the region where $b > 2a$.

Lemma 3.1: *Let (a,b) be in the domain of h and such that $b < 2a$. Then there exists $n \in \mathbb{N}$ such that $h^n(a,b) = (\alpha,\beta)$ satisfies $\beta > 2\alpha$.*

Proof: Letting $(a_0, b_0) = (a,b)$, suppose, on the contrary, that for each $n \geq 0$, $h^n(a,b) = (a_n, b_n)$ satisfies $b_n < 2a_n$. Then, the mapping h takes (a_n, b_n) to $(2a_n - b_n, a_n)$ for all positive n, and we conclude, by induction, that $h^n(a,b) = ((n+1)a - nb, na - (n-1)b)$ for all $n \geq 1$. This implies that for all $n \in \mathbb{N}$, $na - (n-1)b < 2(n+1)a - 2nb$, or

$$2nb - (n-1)b < 2(n+1)a - na.$$

Consequently, $(n+1)b < (n+2)a$, leaving $b < [(n+2)/(n+1)]a$, for all $n \in \mathbb{N}$. Hence, letting $n \to \infty$ implies $b \leq a$, a contradiction. $\square$

We now let $d(a,b)$ denote the (shortest) distance from the point (a,b) to the line $\{(x,y) \in \mathbb{R}^2 : x = y\}$. The following lemma establishes two important facts regarding the effect h has on this distance.

Lemma 3.2: *Let $(a,b) \in \mathbb{R}^2$ with $0 < a < b$ and $a/b \notin \mathbb{Q}$. If (a,b) also satisfies $b < 2a$, then $d(h(a,b)) = d(a,b)$. If(a,b) satisfies $b > 2a$ then $d(h(a,b)) < d(a,b)$.*

Proof: A direct calculation shows that for any point $(r,s) \in \mathbb{R}^2$ we have $d(r,s) = (\sqrt{2}/2)|r-s|$. For$(a,b)$ with $b < 2a$, since $h(a,b) = (2a-b, a)$, we conclude that

$$d(h(a,b)) = \frac{\sqrt{2}}{2}|2a-b-a| = \frac{\sqrt{2}}{2}|b-a| = d(a,b).$$

For (a,b) with $b > 2a$, we again proceed by straight forward calculations, noting that

$$d(h(a,b)) = \frac{\sqrt{2}}{2}|b-2a-b+a| = \frac{\sqrt{2}}{2}a.$$

On the other hand, $d(a,b) = (\sqrt{2}/2)(b-a)$. Since $\sqrt{2}/2a < \sqrt{2}/2(b-a)$ if and only if $b > 2a$, the result follows. $\square$

We are now in a position to prove the following, which completely describes the asymptotic behavior of Ducci sequences which begin with heterogeneous vectors in $\mathbb{R}^3$.

Theorem 3.3: *Suppose $v = (a,b,c)$ is a heterogeneous vector with $a < b < c$. Then, $\lim_{n\to\infty} h^n(g(0, b-a, c-a)) = (0,0)$.*

Proof: As a notational convenience, let $h^n(g(0, b-a, c-a)) = (x_n, y_n)$ for each $n \geq 1$. By Lemma 3.1, we can form asubsequence of $\{(x_n, y_n)\}$ consisting of those terms where $y_n > 2x_n$. We denote this subsequence by $\{(p_n, q_n)\}$. Note that $d(p_n, q_n) > d(p_{n+1}, q_{n+1})$ for all $n \geq 1$ since $d(p_n, q_n) > d(h(p_n, q_n))$ for any $n \in \mathbb{N}$. Consequently, this is a decreasing sequence which is bounded below by zero, and therefore it follows that $\lim\{d(p_n, q_n) - d(p_{n+1}, q_{n+1})\} = 0$.

Referring back to the definition of h, one sees that for any $n \geq 0$, $y_n > y_{n+1}$. Furthermore, the sequence $\{y_n\}$ is also bounded below by zero. Hence, its limit exists, and we let $\lim\{y_n\} = c$. Note that if $c = 0$ then we are done, since for all $n \in \mathbb{N}$, $y_n > x_n$, and we can apply the squeeze theorem to conclude $\lim\{x_n\} = 0$ as well. The remainder of the proof will verify that in fact this must be the case.

Now, for $n \in \mathbb{N}$ given, let i be the smallest positive integer such that $h^i(p_n, q_n) = (P, Q)$ satisfies $Q > 2P$. By definition, it must be that $h^i(p_n, q_n) = (p_{n+1}, q_{n+1})$. If $i = 1$ then

$$d(p_n, q_n) - d(p_{n+1}, q_{n+1}) = \frac{\sqrt{2}}{2}(q_n - 2p_n)$$

follows immediately from Lemma 3.2. If $i > 1$, since $h^l(p_n, q_n)$ satisfies $q_n < 2p_n$ for all $1 \leq l < i$, it follows that

$$d(h(p_n, q_n)) = d(h^2(p_n, q_n)) = \cdots = d(h^i(p_n, q_n)),$$

again by Lemma 3.2. Hence, the same conclusion regarding $d(p_n, q_n) - d(p_{n+1}, q_{n+1})$ applies in this case as well.

Thus, $\lim\{(\sqrt{2}/2)(q_n - 2p_n)\} = 0$, or equivalently, $\lim\{q_n - 2p_n\} = 0$. We know that $\lim\{q_n\} = \lim\{y_n\} = c$, so we conclude $\lim\{p_n\} = c/2$. But recall that $h(p_n, q_n) = (q_n - 2p_n, q_n - p_n)$. Furthermore, the sequence $\{y_n\}$ is decreasing. So $q_{n+1} \leq q_n - p_n$. Thus $\lim\{q_{n+1}\} \leq \lim\{q_n\} - \lim\{p_n\}$, or, $c \leq c - c/2$. It follows that $c \leq 0$. Since $c \geq 0$, we have that $c = 0$, as desired. □

4. CONCLUSIONS AND FUTURE DIRECTIONS

The implication of Theorem 3.3 is that for a given $v \in \mathbb{R}^3$ which is heterogeneous and for any $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that if $n \geq H$, then both entries of $g(f^n(v) - (L, L, L))$ are positive and within ε of zero, where L is the smallest (positive) entry of $f^n(v)$. Thus, each entry of $f^n(v)$ is within ε of zero, and we conclude that $\lim\{f^n(v)\} = (0, 0, 0)$. Our results are then summarized in the following theorem.

Theorem 4.1: *Let $v \in \mathbb{R}^3$. If there exist $\alpha, x \in \mathbb{R}, \alpha \neq 0$, such that $\alpha(v + (x, x, x)) \in \mathbb{Q}^3$, then there exists $n \in \mathbb{N}$ such that $f^n(v)$ is a nontrivial periodic vector, and hence the Ducci sequence $\{f^n(v)\}$ is eventually periodic.*

If, on the other hand, $\alpha(v + (x, x, x)) \notin \mathbb{Q}^3$ for all $\alpha, x \in \mathbb{R}$, $\alpha \neq 0$, then $\{f^n(v)\}$ contains no periodic vectors, but approaches the trivial periodic vector $(0, 0, 0)$ asymptotically.

This result is maybe a bit surprising since it implies that two vectors which are arbitrarily close can have dramatically different asymptotic behavior, with one converging to the zero vector, and the other eventually becoming a nontrivial cyclic vector. In fact, this behavior is distinct from that exhibited by the example in $\mathbb{R}^7$ given by [9] of a vector which approaches, but never becomes, a nontrivial periodic vector. One is then led quite naturally to the question of whether the situations in $\mathbb{R}^5$ or $\mathbb{R}^6$ agree with the present result, or agree with the result of [9]. As observed above, the $\mathbb{R}^4$ case has been resolved by [32], among others.

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