FROM ANDREWS' FORMULA FOR THE FIBONACCI NUMBERS TO THE ROGERS-RAMANUJAN IDENTITIES

Hei-Chi Chan
Department of Mathematical Sciences, University of Illinois at Springfield, Springfield, IL 62703-5407
e-mail: chan.hei-chi@uis.edu
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ABSTRACT

In 1970, Andrews proved a certain polynomial identity (which can be traced back to Schur) and this identity, under appropiate limits, gives the celebrated Rogers-Ramanujan identities. Andrews' method forms the basis for many exciting developments in the last three decades. In this paper, we give an alternative proof of this important result. The key ingredient of our proof is also due to Andrews: it is a technique that Andrews used to prove a new formula for the Fibonacci numbers, dated back to the late 60s.

1. INTRODUCTION

In a recent feature article of this journal [8], G. E. Andrews, the world renowned expert in the work of Srinivasa Ramanujan, points out many intriguing connections between the Fibonacci numbers and the celebrated Rogers-Ramanjuan identities (see [1, 17, 25] for excellent introductions to these identities). The starting point of these beautiful connections is the following representations of the Fibonacci numbers (with $a = 0, 1$)

$$F_{n+1} = \sum_{0\le j\le n} \binom{n-j}{j} = \sum_{j=-\infty}^{\infty} (-1)^j \binom{n+a}{\lfloor \frac{n+3a-5j}{2} \rfloor}, \tag{1}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. The first representation is well-known (e.g., see [12, 23, 29]). The second representation is due to Andrews [2] and forms the basis of the extensive study of generalized Fibonacci numbers [2] (cf. [24, 26]).

To obtain the Rogers-Ramanujan identities, we first q-deform (1) as follows (again, $a = 0, 1$):

$$\sum_{0\le j\le n} q^{j^2+aj} \begin{bmatrix} n-j \\ j \end{bmatrix} = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}-2aj} \begin{bmatrix} n+a \\ \lfloor \frac{n+3a-5j}{2} \rfloor \end{bmatrix}. \tag{2}$$

Note that these formulas are expressed in terms of the q-binomial numbers (or the Gaussian polynomials)

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A \\ \frac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)} & \text{otherwise.} \end{cases}$$

See [1, 9, 19, 27]. (2) can be traced back to Schur [33]. (See also an interesting generalization due to Bressoud [18] and Chapman's alternative proof [20]). In [3], Andrews gives a simpler proof of (2); see also Rademacher's book [31]. To arrive at the Rogers-Ramanujan identities, we simply take the limit $n \to \infty$ on both sides of (2). See [3]; cf. [8, 9]. It should be noted that (2) forms the basis for many exciting developments in the past three decades; e.g., see [4, 5, 6, 7, 10, 11, 13, 14, 15, 16, 18, 21, 22, 28, 30, 32, 34, 35, 36, 37].

Andrews' strategy of proving (2)—a method that is widely used and generalized in later development—is to show that both sides of this equation satisfy the same difference equation and have the same initial values; cf. [3]. On the one hand, it is relatively easy to handle the sum on the left-hand side. On the other hand, it is much more difficult to deal with the sum on the other side; one way to do this (cf. [3]) is to consider separately the case of even n and the case of odd n for the sum on the right-hand side.

The purpose of this article is to show that Andrews' method can be implemented without considering separately the cases of even and of odd n. The key ingredient of our alternate route is due also to Andrews: it is a technique that Andrews used in his 1969 paper published in this journal [2], in which he proves the new representations of F_n in (1). In this approach (Sect. 2), the fifth root of unity plays a key role.

2. THE SET UP AND THE PROOF

Let us denote the left-hand side of (2) by $E_{n+1}(a)$ and the right-hand side by $D_{n+1}(a)$. Our goal is to show that both $E_{n+1}(a)$ and $D_{n+1}(a)$ satisfy the same difference equation and have the same initial values. This implies (2), i.e., $E_{n+1}(a) = D_{n+1}(a)$.

Precisely, we need to prove that (with $a = 0, 1$)

$$E_{n+1}(a) = E_n(a) + q^{n+a-1}E_{n-1}(a), \tag{3}$$
$$E_1(a) = 1, \tag{4}$$
$$E_2(a) = 1, \tag{5}$$

and

$$D_{n+1}(a) = D_n(a) + q^{n+a-1}D_{n-1}(a), \tag{6}$$
$$D_1(a) = 1, \tag{7}$$
$$D_2(a) = 1. \tag{8}$$

Remarks:

- For the proof of (3)-(5), see [3]. Below, we shall focus on the proof of (6)-(8).
- We have indexed the sums $E_{n+1}(a)$ and $D_{n+1}(a)$ so that the initial values shown above are all unity. These match their undeformed partners, namely F_1 and F_2. But note that $E_3(a) = D_3(a)$ and they are given by

$$1 + q^{1+a}. \tag{9}$$

We shall comment on this below.

The initial conditions, (7) and (8), can be easily verified by straightforward computations from the definition of $D_{n+1}(a)$. However, it may be instructive to show (9). Indeed, for $a = 0$, we have

$$\begin{aligned} D_3(0) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} 2 \\ \lfloor \frac{2-5j}{2} \rfloor \end{bmatrix} \\ &= (-1)^0 q^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1-q^2}{1-q} = 1+q. \end{aligned}$$

For the second equality, we have used the fact that only the $j = 0$ term does not vanish.

Similarly, we have for $a = 1$

$$\begin{aligned} D_3(1) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}-2j} \begin{bmatrix} 3 \\ \lfloor \frac{5-5j}{2} \rfloor \end{bmatrix} \\ &= (-1)^0 q^0 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-1)^1 q^{3-2} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ &= \frac{(1-q^3)(1-q^2)}{(1-q^2)(1-q)} - q = 1+q^2. \end{aligned}$$

This time, we have two terms ($j = 0, 1$) contributing to the sum. These verify (9).

Before we prove (6), we need to rewrite $D_{n+1}(a)$ as follows:

Lemma 1:

$$\begin{aligned} D_{n+1}(a) = \frac{(-1)^{n+3a}}{5} \sum_{j=0}^{4} \beta^{j(n+3a)} \Bigg(&\sum_{m \geq 0} q^{f_a(n+3a-2m)} \beta^{-2mj} \begin{bmatrix} n+a \\ m \end{bmatrix} \\ &- \sum_{m \geq 0} q^{f_a(n+3a-2m-1)} \beta^{-(2m+1)j} \begin{bmatrix} n+a \\ m \end{bmatrix} \Bigg), \end{aligned} \tag{10}$$

where

$$f_a(\sigma) = \frac{\sigma(\sigma+1)}{10} - \frac{2}{5} a\,\sigma. \tag{11}$$

Proof: First, we observe that $-1 = (-1)^5$ and so the right-hand side of (2) reads

$$D_{n+1}(a) = \sum_{\sigma \equiv 0 \pmod 5} (-1)^\sigma q^{f_a(\sigma)} \begin{bmatrix} n+a \\ \lfloor \frac{n+3a-\sigma}{2} \rfloor \end{bmatrix}. \tag{12}$$

Next, we want to remove the mod 5 restriction on the sum. Here is the trick used by Andrews [2]. Observe that, with $\beta := e^{i(2\pi/5)}$,

$$\frac{1}{5}\sum_{j=0}^{4}\beta^{j\sigma} = \begin{cases} 1 & \text{if } \sigma \equiv 0 \pmod 5 \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

This allows us to write (12) as

$$D_{n+1}(a) = \frac{1}{5}\sum_{j=0}^{4}\sum_{\sigma=-\infty}^{\infty}(-1)^{\sigma}q^{f_a(\sigma)}\begin{bmatrix} n+a \\ \lfloor \frac{n+3a-\sigma}{2} \rfloor \end{bmatrix}\beta^{j\sigma}. \tag{14}$$

In order to ease the difficulty due to the floor function in the lower argument of the q-binomial numbers, we follow [2]. Define a new valuable λ to replace $n+3a-\sigma$; i.e.,

$$\lambda := n+3a-\sigma.$$

This allows us to write (14) as

$$D_{n+1}(a) = \frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\sum_{\lambda}(-1)^{\lambda}q^{f_a(n+3a-\lambda)}\begin{bmatrix} n+a \\ \lfloor \frac{\lambda}{2} \rfloor \end{bmatrix}\beta^{j\sigma}. \tag{15}$$

Finally, we observe the following property of the floor function:

$$\left\lfloor \frac{2m}{2} \right\rfloor = \left\lfloor \frac{2m+1}{2} \right\rfloor = m.$$

This motivates us to break up the sum in (15) into the sum over the even $\lambda = 2m$ and the sum over the odd $\lambda = 2m+1$. This ultimately removes the floor function and gives (10). □

It will be useful to write down similar expressions for $D_n(a)$ and $D_{n-1}(a)$:

$$D_n(a) = -\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a-1)}\left(\sum_{m\geq 0}q^{f_a(n+3a-2m-1)}\beta^{-2mj}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right.$$
$$\left. - \sum_{m\geq 0}q^{f_a(n+3a-2m-2)}\beta^{-(2m+1)j}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right), \tag{16}$$

$$D_{n-1}(a) = \frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a-2)}\left(\sum_{m\geq 0}q^{f_a(n+3a-2m-2)}\beta^{-2mj}\begin{bmatrix} n+a-2 \\ m \end{bmatrix}\right.$$
$$\left. - \sum_{m\geq 0}q^{f_a(n+3a-2m-3)}\beta^{-(2m+1)j}\begin{bmatrix} n+a-2 \\ m \end{bmatrix}\right). \tag{17}$$

The last ingredient for our proof is the following property of the q-binomial numbers (e.g., see Sect. 3.3 in [1], chap. 7 in [9], chap. 3 in [19] or chap. 6 in [27]):

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} q^{n-m}, \tag{18}$$

$$= \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ m \end{bmatrix} q^{m}, \tag{19}$$

With the above understood, we are ready for the proof of (6).

Proof of (6): Let's look at (6) again (with an emphasis on the first term on the right-hand side)

$$D_{n+1}(a) = D_n(a) + \cdots .$$

This means we need to "produce" $D_n(a)$ from the sum representing $D_{n+1}(a)$. To this end, we do the following. $D_{n+1}(a)$ in (10) consists of two sums over m. Let us apply (18) to replace the q-binomial numbers in the second sum in $D_{n+1}(a)$, and (19) to replace the q-binomial numbers in the first sum in $D_{n+1}(a)$. This gives

$$\begin{aligned} D_{n+1}(a) &= \frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\left(\sum_{m\geq 0} q^{f_a(n+3a-2m)}\beta^{-2mj}\left(\begin{bmatrix} n+a-1 \\ m-1 \end{bmatrix} + \begin{bmatrix} n+a-1 \\ m \end{bmatrix} q^{m}\right)\right. \\ &\left.- \sum_{m\geq 0} q^{f_a(n+3a-2m-1)}\beta^{-(2m+1)j}\left(\begin{bmatrix} n+a-1 \\ m \end{bmatrix} + \begin{bmatrix} n+a-1 \\ m-1 \end{bmatrix} q^{n-m+a}\right)\right) \end{aligned} \tag{20}$$

$$\begin{aligned} &= -\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a-1)}\left(\sum_{m\geq 0} q^{f_a(n+3a-2m-1)}\beta^{-2mj}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right. \\ &\left.- \sum_{m\geq 0} q^{f_a(n+3a-2m)}\beta^{-(2m-1)j}\begin{bmatrix} n+a-1 \\ m-1 \end{bmatrix}\right) \\ &+ \frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\left(\sum_{m\geq 0} q^{f_a(n+3a-2m)+m}\beta^{-2mj}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right. \\ &\left.- \sum_{m\geq 0} q^{f_a(n+3a-2m)+n-m+a}\beta^{-(2m+1)j}\begin{bmatrix} n+a-1 \\ m-1 \end{bmatrix}\right). \end{aligned} \tag{21}$$

Remarks: We organized the second equality as follows. Whenever we use (18) or (19), we end up with two q-binomial numbers, one with an extra q factor and one without. To obtain

(21), we gather into the first big sum all the terms with q-binomial numbers that do not have the extra q factor. This big sum, as we shall see shortly, is $D_n(a)$.

Next, we observe that some terms in (21) involve q-numbers with $m-1$ being their lower arguments. Let us shift the dummy indices to make $m-1$ become m. This makes the first big sum in (21) become $D_n(a)$ (cf. (16)) and (21) now reads

$$\begin{aligned} D_{n+1}(a) &= D_n(a)+ \\ &+\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\left(\sum_{m\geq 0} q^{f_a(n+3a-2m)+m}\beta^{-2mj}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right. \\ &\left.-\sum_{m\geq 0} q^{f_a(n+3a-2m-3)-m+n+a-1}\beta^{-(2m+3)j}\begin{bmatrix} n+a-1 \\ m \end{bmatrix}\right). \end{aligned} \tag{22}$$

It is clear that we need to make the last big sum in (22) become $q^{n+a-1}D_{n-1}(a)$ in (6). To this end, we apply (19) to expand the q-numbers in the second sum (over m). For the other sum, we apply (18). This gives

$$\begin{aligned} D_{n+1}(a) &= D_n(a)+ \\ &+\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m)+m}\beta^{-2mj} \\ &\left(\begin{bmatrix} n+a-2 \\ m \end{bmatrix}+\begin{bmatrix} n+a-2 \\ m-1 \end{bmatrix}q^{n+a-1-m}\right) \\ &-q^{n+a-1}\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m-3)-m}\beta^{-(2m+3)j} \\ &\left(\begin{bmatrix} n+a-2 \\ m-1 \end{bmatrix}+\begin{bmatrix} n+a-2 \\ m \end{bmatrix}q^{m}\right) \end{aligned} \tag{23}$$

On the right-hand side of (23), focus on the terms that have the extra q factors after the application of (18) and (19). Collect them together and this gives $q^{n+a-1}D_{n-1}(a)$:

$$\begin{aligned} &\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m)+m}\beta^{-2mj}\begin{bmatrix} n+a-2 \\ m-1 \end{bmatrix}q^{n+a-1-m} \\ &-q^{n+a-1}\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m-3)-m}\beta^{-(2m+3)j}\begin{bmatrix} n+a-2 \\ m \end{bmatrix}q^{m} \\ &=q^{n+a-1}\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a-2)}\sum_{m\geq 0} q^{f_a(n+3a-2m-2)}\beta^{-2mj}\begin{bmatrix} n+a-2 \\ m \end{bmatrix} \\ &-q^{n+a-1}\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a-2)}\sum_{m\geq 0} q^{f_a(n+3a-2m-3)}\beta^{-(2m+1)j}\begin{bmatrix} n+a-2 \\ m \end{bmatrix} \\ &=q^{n+a-1}D_{n-1}(a). \end{aligned}$$

Above, we shifted the dummy indices from $m-1$ to m to obtain the first equality. Then, by comparing to (17), we obtain the second equality.

To proceed, we note that there is another sum (over m) in (23) that has q-binomial numbers with the lower argument being $m-1$ (but does not have the extra q factor due to the application of (19)). Again, we need to shift the dummy indices in this sum as follows:

$$
\begin{aligned}
&-\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m-3)+n+a-1-m}\beta^{-(2m+3)j}\begin{bmatrix} n+a-2 \\ m-1 \end{bmatrix} \\
&= -\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m-5)+n+a-2-m}\beta^{-(2m+5)j}\begin{bmatrix} n+a-2 \\ m \end{bmatrix} \\
&= -\frac{(-1)^{n+3a}}{5}\sum_{j=0}^{4}\beta^{j(n+3a)}\sum_{m\geq 0} q^{f_a(n+3a-2m-5)+n+a-2-m}\beta^{-2mj}\begin{bmatrix} n+a-2 \\ m \end{bmatrix}.
\end{aligned}
$$

Note that the difference between the second and the third lines lies in the exponent of β: the "5" disappears in the third line. This is because $\beta = e^{i(2\pi/5)}$ and therefore $\beta^5 = 1$.

The foregoing discussion allows us to write (23) as

$$D_{n+1}(a) = D_n(a) + q^{n+a-1}D_{n-1}(a) + \Delta_n(a),$$

where

$$
\begin{aligned}
\Delta_n(a) = \frac{(-1)^{n+3a}}{5}\sum_{j,m}\beta^{j(n+3a-2m)}\begin{bmatrix} n+a-2 \\ m \end{bmatrix} \\
\left(q^{f_a(n+3a-2m)+m} - q^{f_a(n+3a-2m-5)+n+a-2-m}\right).
\end{aligned}
$$

If $\Delta_n(a)$ is zero, we are done. And this is easy to verify. From the definition of $f_a(\sigma)$ (cf. (11)), a direct calculation shows that

$$f_a(n+3a-2m)+m = f_a(n+3a-2m-5)+n+a-2-m$$

and so

$$q^{f_a(n+3a-2m)+m} - q^{f_a(n+3a-2m-5)+n+a-2-m} = 0.$$

This makes $\Delta_n(a)$ vanish and the desired result is proven. □

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