CONSECUTIVE ZECKENDORF-NIVEN AND LAZY-FIBONACCI-NIVEN NUMBERS

Helen G. Grundman

Department of Mathematics, Bryn Mawr College, 101 N. Merion Avenue, Bryn Mawr, PA 19010-2899

(Submitted November 2006)

ABSTRACT

A positive integer n is a Zeckendorf-Niven number if the sum of the coefficients of its Zeckendorf expansion is a divisor of n. Lazy-Fibonacci-Niven numbers are defined analogously. It is shown that any sequence of consecutive Zeckendorf-Niven numbers greater than six is of length at most four and that there exist infinitely many such sequences of length four. Interestingly, the same maximal length holds for sequences of lazy-Fibonacci-Niven numbers, as is also shown in this paper.

1. INTRODUCTION

Given a positive integer n, there exists at least one sequence $\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots$ such that $\varepsilon_i \in \{0,1\}$, for each $i \geq 2$, and

$$n = \sum_{i \geq 2} \varepsilon_i F_i.$$

Considering the set of all such sequences for a fixed n in reverse lexicographic order (or, equivalently, considering all such expressions, $(\ldots, \varepsilon_4, \varepsilon_3, \varepsilon_2)$ in lexicographic order), the expansion of n corresponding to the maximal sequence is called the *Zeckendorf expansion* [3] and that corresponding to the minimal sequence is called the *lazy-Fibonacci expansion.* See Table 1 for examples.

Number	Zeckendorf	$S_Z(n)$	Lazy-Fibonacci	$S_L(n)$
1	F_2	1	F_2	1
2	F_3	1	F_3	1
3	F_4	1	$F_3 + F_2$	2
4	$F_4 + F_2$	2	$F_4 + F_2$	2
5	F_5	1	$F_4 + F_3$	2
6	$F_5 + F_2$	2	$F_4 + F_3 + F_2$	3
7	$F_5 + F_3$	2	$F_5 + F_3$	2
8	F_6	1	$F_5 + F_3 + F_2$	3
9	$F_6 + F_2$	2	$F_5 + F_4 + F_2$	3
10	$F_6 + F_3$	2	$F_5 + F_4 + F_3$	3

Table 1. Zeckendorf and Lazy-Fibonacci Expansions

For each $n \in \mathbb{Z}^+$, let $S_Z(n)$, and $S_L(n)$, denote the sum of the coefficients, $\varepsilon_i(n)$, of the Zeckendorf expansion of n, and of the lazy-Fibonacci expansion of n. Steiner [2] studied the functions $S_Z(n)$ and $S_L(n)$ (using different notation) and described their joint distribution.

A positive integer is a Niven number if it is divisible by the sum of its digits. We say that $n \in \mathbb{Z}^+$ is a *Zeckendorf-Niven* number (or ZN) if $S_Z(n)$ is a divisor of n and a *lazy-Fibonacci-Niven* number (or LFN) if $S_L(n)$ is a divisor of n. For example, as seen in Table 1, the numbers 1–6 are all ZN, but 7 is not; 1 and 2 are LFN, but 3 is not.

Ray and Cooper [1] proved that the natural density of Zeckendorf-Niven numbers (and of the more general k-Zeckendorf-Niven numbers) is zero. The purpose of the next two sections is to consider the possible lengths of sequences of consecutive Zeckendorf-Niven numbers and lazy-Fibonacci-Niven numbers. In the final section we raise some further questions, some of which we then answer.

2. ZECKENDORF-NIVEN NUMBERS

In this section we consider sequences of consecutive Zeckendorf-Niven numbers. Let $n \in \mathbb{Z}^+$ and let $\zeta_i(n)$ denote the coefficients in the Zeckendorf-Niven expansion of n,

$$n = \sum_{i \geq 2} \zeta_i(n) F_i.$$

Further, let

$$z_5(n) = \sum_{i=2}^{5} \zeta_i(n) F_i.$$

Lemma 1: *Let $n \geq 8$. If $z_5(n) = 1$, 2, or 6, then n and $n+1$ are not both ZN.*

Proof: Note that $1 = F_2$, $2 = F_3$, and $6 = F_5 + F_2$. So if $z_5(n) = 1$, 2, or 6, then $S_Z(n) = S_Z(n+1)$. If both n and $n+1$ are ZN, then $S_Z(n)$ divides each and therefore $S_Z(n)$ divides their difference, $(n+1) - n = 1$. So $S_Z(n) = 1$ which is impossible if $n \geq 8$ and $z_5(n) \geq 1$. $\square$

Theorem 2: *The only sequences of five or more consecutive Zeckendorf-Niven numbers are subsequences of:* $1, 2, 3, 4, 5, 6$.

Proof: Suppose n, $n+1$, $n+2$, $n+3$, and $n+4$ are all ZN with $n \geq 3$. Since 7 is not ZN, we can assume that $n \geq 8$.

By Lemma 1, $z_5(n) \neq 1$, 2, or 6. If $z_5(n) = 0$, then $z_5(n+1) = 1$, contradicting the lemma. Similarly, if $z_5(n) = 5$, then $z_5(n+1) = 6$ and if $z_5(n) = 7$, then $z_5(n+2) = 1$, again yielding contradictions.

Thus $z_5(n) = 3$ or 4. For $z_5(n) = 3$, if $\zeta_6(n) = 0$, then $z_5(n+3) = 6$ and if $\zeta_6(n) = 1$, then $z_5(n+3) = 1$. For $z_5(n) = 4$, if $\zeta_6(n) = 0$, $z_5(n+2) = 6$ and if $\zeta_6(n) = 1$, $z_5(n+2) = 1$. Hence each case contradicts Lemma 1. $\square$

Note that the above proof shows that any sequence of four consecutive Zeckendorf-Niven numbers must begin with an integer n for which $z_5(n) = 3$. We now show that not only does such a sequence exist, but that there are infinitely many of them.

Theorem 3: *There exist infinitely many sequences of four consecutive Zeckendorf-Niven numbers.*

Proof: Let

$$n = F_{120k-6} + F_8 + F_6 + F_4,$$

with $k \in \mathbb{Z}^+$. It is easily confirmed that the values of the Fibonacci numbers modulo 3, 4, and 5 are in cycles of lengths 8, 6, and 20, respectively. Thus, since $F_8 + F_6 + F_4 = 32$,

$$\begin{aligned} n &\equiv F_2 + 32 \equiv 0 \pmod 3; \\ n &\equiv F_0 + 32 \equiv 0 \pmod 4; \\ n &\equiv F_{14} + 32 \equiv 4 \pmod 5. \end{aligned} \tag{1}$$

We will show that for every value of $k \geq 1$, $n, n+1, n+2, n+3$ is a sequence of four consecutive ZN numbers. First, by inspection, $S_Z(n) = 4$. Further, $S_Z(n+1) = 5$, $S_L(n+2) = 2$, and $S_L(n+3) = 3$. Therefore, by equation (1), $S_Z(n+i) | (n+i)$, for $0 \leq i \leq 3$. $\square$

3. LAZY-FIBONACCI-NIVEN NUMBERS

We now consider sequences of consecutive lazy-Fibonacci-Niven numbers. In analogy to the previous section, let $n \in \mathbb{Z}^+$ and let $\lambda_i(n)$ denote the coefficients in the lazy-Fibonacci-Niven expansion of n,

$$n = \sum_{i \geq 2} \lambda_i(n) F_i.$$

Let

$$\ell_4(n) = \sum_{i=2}^{4} \lambda_i(n) F_i.$$

Lemma 4: *Let $n \geq 6$. If $\ell_4(n) = 1$, 3, or 4, then n and $n+1$ are not both LFN.*

The proof follows the same reasoning as the proof of Lemma 1 and so is omitted.

Theorem 5: *There are no sequences of five consecutive lazy-Fibonacci-Niven numbers.*

Proof: Suppose that $n \in \mathbb{Z}^+$ such that n, $n+1$, $n+2$, $n+3$, and $n+4$ are all LFN. From Table 1, we can assume that $n > 10$.

Note that if $\ell_4(n) = 0$, then $\ell_4(n+1) = 1$ and if $\ell_4(n) = 2$, then $\ell_4(n+1) = 3$. Thus by Lemma 4, $\ell_4(n) \notin \{0, 1, 2, 3, 4\}$, that is, $\ell_4(n) = 5$ or 6.

Let $m \in \{n, n+1\}$ be such that $\ell_4(m) = 6$. For either value of m, all three of $m+1$, $m+2$, and $m+3$ are LFN. Let $j \geq 5$ be the smallest index such that $\lambda_j(m) = 0$. There are two cases to consider. If j is even, then $\ell_4(m+1) = 4$ and so by Lemma 4, $m+2$ is not LFN. If j is odd, then $\ell_4(m+1) = 2$. But then $\ell_4(m+2) = 3$ and by Lemma 4, $m+3$ is not LFN. So each case leads to a contradiction. $\square$

We now show that the bound given in Theorem 5 is the best possible.

Theorem 6: *There exists a sequence of four consecutive lazy-Fibonacci-Niven numbers.*

Proof: Let $n = 3{,}674{,}769$. Then we have

$$\begin{aligned}
n &= F_{32} + F_{30} + F_{28} + F_{27} + F_{25} + F_{24} + F_{22} + F_{20} + F_{18} + F_{16} + \\
&\quad F_{14} + F_{12} + F_{11} + F_{10} + F_9 + F_8 + F_7 + F_6 + F_5 + F_4 + F_3 \\
n+1 &= F_{32} + F_{30} + F_{28} + F_{27} + F_{25} + F_{24} + F_{22} + F_{20} + F_{18} + F_{16} + \\
&\quad F_{14} + F_{12} + F_{11} + F_{10} + F_9 + F_8 + F_7 + F_6 + F_5 + F_4 + F_3 + F_2 \\
n+2 &= F_{32} + F_{30} + F_{28} + F_{27} + F_{25} + F_{24} + F_{22} + F_{20} + F_{18} + F_{16} + \\
&\quad F_{14} + F_{13} + F_{11} + F_9 + F_7 + F_5 + F_3 \\
n+3 &= F_{32} + F_{30} + F_{28} + F_{27} + F_{25} + F_{24} + F_{22} + F_{20} + F_{18} + F_{16} + \\
&\quad F_{14} + F_{13} + F_{11} + F_9 + F_7 + F_5 + F_3 + F_2.
\end{aligned}$$

A straightforward check shows that these are all LFN numbers. □

We note that a direct computer search shows that 3,674,769 is, in fact, the smallest number initiating a sequence of four consecutive lazy-Fibonacci-Niven numbers.

4. FURTHER QUESTIONS

Finally, we consider integers satisfying a much weaker requirement. We call a positive integer n a *weak-Fibonacci-Niven* number (or WFN) if there is *any* expansion $n = \sum_{i\geq 2} \varepsilon_i F_i$ with $\varepsilon_i \in \{0, 1\}$, for all i, for which $\sum_{i\geq 2} \varepsilon_i$ is a divisor of n.

Certainly, from Table 1, the numbers 1–6, 8, 9, and 10 are all WFN. The number 7 is not WFN since it has only one expansion which has coefficient sum 2.

Is it true that, as with the numbers 1–10, all weak-Fibonacci-Niven numbers are either ZN or LFN? The answer is no. For example, $40 = F_8 + F_7 + F_5 + F_2$ and so is WFN. Yet $S_Z(40) = 3$ and $S_L(40) = 6$, so 40 is neither ZN nor LFN.

Do there exist more than four consecutive weak-Fibonacci-Niven numbers (other than the subsequences of 1–6)? If so, are there infinitely many such sequences? Is there a maximal length of sequences of consecutive weak-Fibonacci-Niven numbers, and if so, what is that length?

Certainly there exist more than four consecutive weak-Fibonacci-Niven numbers larger than 6. For example, it's easily checked that 12–16 are all WFN. Below we show that there are infinitely many sequences of six consecutive weak-Fibonacci-Niven numbers. We leave the questions about the maximal possible length of such sequences for future research.

Theorem 7: *There exist infinitely many sequences of six consecutive weak-Fibonacci-Niven numbers.*

Proof: Let

$$n = F_{240k} + F_{14} + F_9 = F_{240k} + 411,$$

with $k \in \mathbb{Z}^+$. The values of the Fibonacci numbers modulo 3, 4, 5, and 7 are in cycles of lengths 8, 6, 20, and 16 respectively. Thus, $F_{240k} \equiv 0 \pmod{420}$.

So, $3|n$, $4|(n+1)$, $7|(n+2)$, $6|(n+3)$, $5|(n+4)$, and $4|(n+5)$. Further,

$$\begin{aligned}
n &= F_{240k} + F_{14} + F_9 \\
n+1 &= F_{240k} + F_{14} + F_9 + F_2 \\
n+2 &= F_{240k} + F_{13} + F_{12} + F_8 + F_6 + F_5 + F_3 \\
n+3 &= F_{240k} + F_{14} + F_8 + F_6 + F_5 + F_4 \\
n+4 &= F_{240k} + F_{14} + F_9 + F_4 + F_2 \\
n+5 &= F_{240k} + F_{14} + F_9 + F_5.
\end{aligned}$$

Thus for each $k \in \mathbb{Z}^+$, n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$ is a sequence of six consecutive weak-Fibonacci-Niven numbers. $\square$

REFERENCES

[1] A. Ray and C. Cooper. "On the Natural Density of the k-Zeckendorf Niven Numbers." *J. Inst. Math. Comput. Sci. Math.* **19** (2006): 83-98.

[2] W. Steiner. "The Joint Distribution of Greedy and Lazy Fibonacci Expansions." *Fibonacci Quarterly* **43** (2005): 60-69.

[3] E. Zeckendorf. "Représentation des Nombres Naturels Par Uhne Somme de Nombres de Fibonacci ou de Nombres de Lucas." *Bull. Soc. Roy. Sci. Liège* **41** (1972): 179-182.

AMS Classification Numbers: 11A63, 11B39

✠ ✠ ✠