AN EXTENSION OF THE GCD STAR OF DAVID THEOREM

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ABSTRACT

Gould's Star of David Theorem is a remarkable result that has generated much interest and a substantial literature since it first appeared in 1971. Extensions of the theorem have been proved for some hexagons of coefficients in Pascal's triangle (and generalizations thereof) having an even number of coefficients per side and for some with an odd number of coefficients per side, as well as for some other configurations. Almost all of these results are for small configurations involving only a limited number of coefficients per side. Here we prove a similar result for a hexagon with an arbitrarily large number of entries on two parallel sides and two coefficients on each of the other four sides.

1. INTRODUCTION

The surprising GCD Star of David Theorem was conjectured by Gould [4] in 1972 and first proved by Hoggatt and Hillman [5]. During the intervening years a substantial number of alternative proofs and interesting generalizations and related results have been added to the literature. The outstanding conjecture (see [9]) is that if $S = S_1 \cup S_2$ denotes the set of binomial coefficients numbered consecutively around an arbitrary "hexagon" of binomial coefficients with sides along the horizontal rows and main diagonals of Pascal's triangle and with an even number of coefficients per side, and if S_1 denotes the set of odd numbered coefficients and S_2 denotes the set of even numbered coefficients, then $\mathrm{GCD}(S_1) = \mathrm{GCD}(S_2)$ where $\mathrm{GCD}(S_i)$ denotes the greatest common divisor of the elements of S_i, $i = 1, 2$.

Let $(m, 2, 2, m, 2, 2)$ denote a convex hexagon of binomial coefficients with m adjacent coefficients along the top and bottom rows and two adjacent coefficients along each of the other four sides as in Figure 1. The equality of the GCDs of

$$\begin{array}{ccccccc} & \binom{k}{r} & \binom{k}{r+1} & \binom{k}{r+2} & \cdots & \binom{k}{r+m-2} & \binom{k}{r+m-1} & \\ \binom{k+1}{r} & & & & & & & \binom{k+1}{r+m} \\ & \binom{k+2}{r+1} & \binom{k+2}{r+2} & \binom{k+2}{r+3} & \cdots & \binom{k+2}{r+m-1} & \binom{k+2}{r+m} & \end{array}$$

Figure 1

the two sets S_1 and S_2 of odd and even numbered coefficients numbered consecutively around the $(2m, 2, 2, 2m, 2, 2)$ hexagon situated arbitrarily in Pascal's triangle and for the same hexagons rotated 120° and 240° was proved by Korntved [6] and also follows immediately from Lemmas 1 and 2 below. In the present paper, we consider similarly placed

$(2m+1, 2, 2, 2m+1, 2, 2)$ hexagons with vertices $a_1, a_2, a_3, \ldots, a_{2m+2}, b_1, b_2, b_3, \ldots, b_{2m+2}$ as indicated in Figure 2. With $a_1 = \binom{n}{r}$ and $s = n - r$. We set $S_a = \{a_i \,|\, 1 \leq i \leq 2m+2\}$ and $S_b = \{b_i \,|\, 1 \leq i \leq 2m+2\}$. Our purpose is to show that GCD(S_a)=t·GCD(S_b) with $t = 1$ or $t = 2$ and to specify when $t = 1$ and $t = 2$.

$$\begin{array}{ccccccccccccc} & a_1 & & b_1 & & a_2 & & b_2 & \cdots & a_m & b_m & a_{m+1} & \\ b_{2m+2} & & x & & y & & z & \cdots & & & & & b_{m+1} \\ & a_{2m+2} & & b_{2m+1} & & a_{2m+1} & & b_{2m} & \cdots & a_{m+3} & b_{m+2} & a_{m+2} & \end{array}$$

Figure 2.

2. NOTATION AND PRELIMINARY RESULTS

Let p be a prime and let $v = v_p(\alpha)$ denote the $p-$adic valuation of α. If α is a rational number, then $\alpha = p^v a/b$ where $(a,p) = (b,p) = 1$. If $\alpha = n$ is an integer then $p^v || n$; i.e. $p^v | n$ and $p^{v+1} \not| n$. Moreover, it is clear that

$$v(1) = 0 \tag{1}$$
$$v(\alpha\beta) = v(\alpha) + v(\beta) \tag{2}$$
$$v(\alpha/\beta) = v(\alpha) - v(\beta) \tag{3}$$
$$v(\alpha \pm \beta) \geq \min(v(\alpha), v(\beta)) \;\; \forall \alpha, \beta \tag{4}$$
$$v(\alpha \pm \beta) = \min(v(\alpha), v(\beta)) \quad \text{if } v(\alpha) \neq v(\beta)\,. \tag{5}$$

Also,

$$\mathrm{GCD}(m_1, m_2, \ldots, m_k) = \prod_{p|m} p^{\min(v(m_1), \ldots, v(m_k))}\,. \tag{6}$$

where $m = \prod_{i=1}^{k} m_i$. Finally, the following two easily proved lemmas (see[2]) are needed.

Lemma 1: In the three cases shown, if x, y, z are adjacent coefficients in Pascal's

$$\begin{array}{ccc} x & & y \\ & z & \end{array}\;,\qquad \begin{array}{ccc} x & & z \\ & y & \end{array}\;,\qquad \begin{array}{ccc} z & & x \\ & y & \end{array}$$

triangle, then $\mathrm{GCD}(x, y) = \mathrm{GCD}(x, y, z)$.

Lemma 2: In the two cases shown, if x, y, z, w are coefficients in Pascal's triangle with x, y, z adjacent to w,

$$\begin{array}{ccc} & & y \\ x & w & \\ & & z \end{array}\;,\qquad \begin{array}{ccc} y & & \\ & w & x \\ z & & \end{array}$$

then $\mathrm{GCD}(x, y, z) = \mathrm{GCD}(x, y, z, w)$.

3. THE MAIN RESULTS

Theorem 1: In the hexagon of Figure 2, $\mathrm{GCD}(S_a) = t\cdot\, \mathrm{GCD}(S_b)$ with $t = 2$, if and only if r and s are odd and $v_2(r + 2h - 1) = v_2(s - 2h + 3)$ for $1 \leq h \leq m+1$. Otherwise, $t = 1$.

Proof: We first show that $\mathrm{GCD}(S_b) \leq \mathrm{GCD}(S_a)$. Suppose $d = \mathrm{GCD}(S_b)$. Then $d|b_i$, $1 \leq i \leq 2m+2$. But then, by Lemma 2, $d|x$ in Figure 2. Now, using Lemma 1 repeatedly, we obtain in order $d|a_1$, $d|a_{2m+2}$, $d|y$, $d|a_2$, $d|z$, $d|a_{2m+1}$ and so on. Then, finally, $d|a_i$, $1 \leq i \leq 2m+2$ and hence $\mathrm{GCD}(S_b)|\mathrm{GCD}(S_a)$ and $\mathrm{GCD}(S_b) \leq \mathrm{GCD}(S_a)$.

Next let p denote a prime and let $v_p(T) = \min\{v_p(t) \,|\, t \in T\}$. Assume that $v_p(S_b) < v_p(S_a)$. Let $e = v_p(S_a \cup \{b_1\})$. Then $p^e|a_i$, $1 \leq i \leq 2m+2$, and $p^e|b_1$. Referring to Figure 2 and using Lemma 1, we have successively that $p^e|x$, $p^e|b_{2m+2}$, $p^e|y$, $p^e|b_{2m+1}$, $p^e|z$, $p^e|b_2, \ldots, p^e|b_{2m}$. Thus, $p^e|b_i$ for $1 \leq i \leq 2m+2$, and

$$v_p(S_a \cup \{b_1\}) \leq v_p(S_a \cup S_b) \leq v_p(S_b)\,.$$

But, by assumption, $v_p(S_b) < v_p(S_a)$, and hence $v_p(b_1) = v_p(S_b)$. Since $a_1 = \binom{n}{r}$, it follows that $a_1 = \frac{r+1}{s}b_1$, $a_2 = \frac{s-1}{r+2}b_1$, and $a_{2m+2} = \frac{(n+1)(n+2)}{s(s+1)}b_1$. But then, since $v_p(b_1) = v_p(S_b) < v_p(S_a)$, it follows that $v_p(\frac{r+1}{s}) > 0$, $v_p(\frac{s-1}{r+2}) > 0$, and $v_p(\frac{(n+1)(n+2)}{s(s+1)}) > 0$. Hence, by (3), $v_p(s) < v_p(r+1)$ and $v_p(r+2) < v_p(s-1)$, and $p|s-1$, $p|r+1$, $p \nmid s$, $p|n$, $p \nmid n+1$. This implies that

$$v_p\left(\frac{(n+1)(n+2)}{s(s+1)}\right) = v_p\left(\frac{n+2}{s+1}\right) > 0,$$

and hence $v_p(s+1) < v_p(n+2)$. Therefore, $p|n+2$. But $p|n$, so $p = 2$. Therefore, $v_p(S_b) \geq v_p(S_a)$ if p is odd. But from above $\mathrm{GCD}(S_b) \leq \mathrm{GCD}(S_a)$. Thus, $v_p(S_b) \leq v_p(S_a)$, and so $v_p(S_b) = v_p(S_a)$ for p odd. It follows, therefore, that $\mathrm{GCD}(S_a) = t \cdot \mathrm{GCD}(S_b)$ with $t = 2^k$, $k \geq 0$.

We now take $v = v_2$ and continue to assume that $v(S_b) < v(S_a)$. Then, since $v(b_1) = v(S_b)$, it follows that $v(b_1) < v(a_1)$ and $v(b_1) < v(a_2)$, and this implies that $2|r+1$ and $2|s-1$. But then $2|n$; so n is even and r and s are odd. If $t \geq 4$ then $4|r+1$, $4|s-1$ and $4|n$ so that $2||s+1$ and $2||n+2$. Therefore, since s and $n+1$ are odd, $v(a_{2m+2}) = v(b_1) = v(S_b)$ as above, and this is a contradiction. Thus $t = 2$. Moreover, we observe that the above argument can be repeated for each b_i, $2 \leq i \leq m$, with only minor changes, leading in each case to the conclusion that, if $\mathrm{GCD}(S_b) < \mathrm{GCD}(S_a)$ then $\mathrm{GCD}(S_a) = 2 \cdot \mathrm{GCD}(S_b)$. We need only consider $a_{m+2}, a_{m+3}, \ldots, a_{2m+2}$ to see when this is so. In fact, we determine conditions under which $v(a_h) > v(S_b)$ for $m+2 \leq h \leq 2m+2$. Since $a_{2m+2} = \frac{(n+1)(n+2)}{s(s+1)}b_1$ and $v(b_1) = v(S_b)$, $v(a_{2m+2}) > v(S_b)$ iff $v(a_{2m+2}) > v(b_1)$ iff $v(n+2) > v(s+1)$ iff $v(s+1) = v(r+1)$ by (5). Similarly $a_{2m+1} = \frac{(n+1)(n+2)}{(s-1)(s-2)}b_2$, and $v(b_2) = v(S_b)$, $v(a_{2m+1}) > v(S_b)$ iff $v(a_{2m+1}) > v(b_2)$ iff $v(n+2) > v(s-1)$ iff $v(s-1) = v(r+3)$. Continuing in this way, we have that $v(a_{2m+3-h}) > v(S_b)$ iff $v(a_{2m+3-h}) > v(b_h)$ iff $v(s-2h+3) = v(r+2h-1)$ for $3 \leq h \leq m$. That this is also true for $h = m+1$ follows from a similar argument involving a_{m+2} and b_{m+1}. Therefore, in order that $\mathrm{GCD}(S_b)$=2 $\mathrm{GCD}(S_a)$, it is necessary that $v(s-2h+3) = v(r+2h-1)$ for $1 \leq h \leq m+1$ as claimed.

We now show that these conditions are sufficient to give $t = 2$. Since $a_1 = \frac{r+1}{s}b_1$ and r and s are odd, it follows that $v(a_1) > v(b_1) \geq v(S_b)$. Moreover, $a_h = \frac{s-2h+3}{r+2h-2} \cdot b_{h-1}$ for $2 \leq h \leq m+1$. And again, r and s odd imply that

$$v(a_h) > v(b_{h-1}) \geq v(S_b).$$

Therefore, $v(a_h) > v(S_b)$ for $1 \leq h \leq m+1$. Similarly, as noted above, $a_{2m+2} = \frac{(n+1)(n+2)}{s(s+1)} \cdot b_1$, so $v(a_{2m+2}) = v(\frac{(n+1)(n+2)}{s(s+1)} \cdot b_1) = v(\frac{n+2}{s+1} \cdot b_1)$ since $n+1$ and s are odd. But $n+2 = (r+1)$ $+(s+1)$, and the first of the above conditions with $h=1$ gives $v(s+1) = v(r+1)$. This implies that $v(n+2) > v(s+1)$ and hence that $v(a_{2m+2}) > v(b_1) \geq v(S_b)$. Similarly, $a_{2m+3-h} = \frac{(n+1)(n+2)}{(r+2h-2)(r+2h-1)} \cdot b_{h-1}$, so $v(a_{2m+3-h}) = v(\frac{(n+1)(n+2)}{(r+2h-2)(r+2h-1)} \cdot b_{h-1}) = v(\frac{n+2}{r+2h-1} \cdot b_{h-1})$ since $n+1$ and $r+2h-2$ are odd. But $n+2 = (s-2h+3)+(r+2h-1)$ and $v(s-2h+3) = v(r+2h-1)$, so $v(n+2) > v(r+2h-1)$ and $v(a_{2m+3-h}) > v(b_{h-1}) \geq v(S_b)$ for $2 \leq h \leq m+1$. Thus, it follows that $v(S_a) = \min\{v(a_i)\,|\,1 \leq i \leq 2m+2\} > v(S_b)$ and that $t=2$ as claimed. This completes the proof.

Now, for subsequent use, consider the sequence V of 2-adic valuations of the even positive intergers as shown here.

E	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
V	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5

34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64...
1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	6...

It is easy to see that the following assertions regarding V are true.

- The integers e_i for which $v(e_i) = k$ (i.e., for which $e_i = t \cdot 2^k$ with t odd) are just the integers congruent to 2^k modulo 2^{k+1}.
- The $2^k - 1$ consecutive entries in V centered at $v(e_i) = k$ form a finite symmetric subsequence of V with a single maximum at $v(e_i)$.
- The symmetric subsequence of V centered at $v(e_i) = k$ is precisely the same as the symmetric sequence of length $2^k - 1$ starting at $v(2) = 1$ and centered at $v(2^k)$.

Now it turns out that the conditions of Theorem 1 can be given in a somewhat more transparent form. Before doing so it will be useful to consider a couple of examples. As above, we here take $v = v_2$.

Example 1: m=1

$$\begin{array}{ccccc} & a_1 & b_1 & a_2 & \\ b_4 & & & & b_2 \\ & a_4 & b_3 & a_3 & \end{array}$$

In this case, we have from Theorem 1 that, for $\text{GCD}(S_a) = 2{\cdot}\text{GCD}(S_b)$, it is necessary and sufficient that r and s be odd and $v(r+2h-1) = v(s-2h+3)$ for $1 \leq h \leq 2$; i.e., that $v(r+1) = v(s+1)$ and $v(r+3) = v(s-1)$. This can be accomplished in one of two ways. One or the other of the sequences

$$\begin{array}{ccc} & v(r+1), & v(r+3) \\ & \| & \\ v(s-1), & v(s+1) & \end{array}$$

or

$$\begin{array}{lll} v(r+1), & v(r+3) & \\ & \parallel & \\ & v(s-1), & v(s+1), \end{array}$$

where the middle terms in each case are the equalities determined by $h = 1$ and $h = 2$ respectively, must be symmetric. Since both are of length three and the first symmetric subsequence of V of length 3 is centered at $v(2^2)$, it follows from the preceding comments about V that the first case can be accomplished if and only if

$$r+1 \equiv s+1 \equiv 2^k \pmod{2^{k+1}},\ k \geq 2,$$

and the second can be accomplished if and only if

$$r+3 \equiv s-1 \equiv 2^k \pmod{2^{k+1}},\ k \geq 2.$$

Example 2: m=2

$$\begin{array}{ccccccc} & a_1 & b_1 & a_2 & b_2 & a_3 & \\ b_6 & & & & & & b_3 \\ & a_6 & b_5 & a_5 & b_4 & a_4 & \end{array}$$

Here we must have $v(r+1) = v(s+1)$, $v(r+3) = v(s-1)$ and $v(r+5) = v(s-3)$. Hence the possibilities are that one of the sequences

$$\begin{array}{lllll} & & v(r+1), & v(r+3), & v(r+5) \\ & & \parallel & & \\ v(s-3), & v(s-1), & v(s+1) & & \end{array}$$

or

$$\begin{array}{lll} v(r+1), & v(r+3), & v(r+5) \\ \parallel & \parallel & \parallel \\ v(s-3), & v(s-1), & v(s+1) \end{array}$$

or

$$\begin{array}{lllll} v(r+1), & v(r+3), & v(r+5) & & \\ & & \parallel & & \\ & & v(s-3), & v(s-1), & v(s+1) \end{array}$$

where the middle terms in each case are the equalities determined by $h = 1$, $h = 2$, and $h = 3$ respectively, must be symmetric. Since the first symmetric subsequences of V of length 5 and

3 respectively are centered at $v(2^3)$ and $v(2^2)$, it follows that $\mathrm{GCD}(S_a) = 2{\cdot}\mathrm{GCD}(S_b)$ if and only if

$$\begin{array}{lllll} r+1 & \equiv s+1 & \equiv 2^k & \pmod{2^{k+1}}, & k>2, \\ r+3 & \equiv s-1 & \equiv 2^k & \pmod{2^{k+1}}, & k\geq 2, \quad \text{or} \\ r+5 & \equiv s-3 & \equiv 2^k & \pmod{2^{k+1}}, & k>2 \end{array}$$

hold.

We now state the general theorem.

Theorem 2: Let q be the greatest integer such that $2^q \leq m+1$ and let $j = m+1-2^q$. For the hexagon of Figure 2, $\mathrm{GCD}(S_a) = t{\cdot}\mathrm{GCD}(S_b)$ with $t = 1$ or 2. In order that $t = 2$ it is necessary and sufficient that

$$r+2h-1 \equiv s-2h+3 \equiv 2^k \pmod{2^{k+1}}$$

with $k > q+1$ for $1 \leq h \leq j$ or $m+2-j \leq h \leq m+1$, and $k \geq q+1$ for $j < h < m+2-j$. In case $j = 0$, all congruences hold for $k \geq q+1$.

Proof: That $t = 1$ or $t = 2$ is shown in Theorem 1. We now consider when $t = 2$. Clearly, any positive integer m can be expressed uniquely in the form stated in the theorem. Moreover, as in the examples, the equalities of Theorem 1 give rise to $m+1$ symmetric subsequences of V of length

$$\ell_h = \begin{cases} 2m+1-2(h-1), & 1 \leq h \leq \frac{m+1}{2} \\ 2h-1, & \frac{m+1}{2} < h \leq m+1\,. \end{cases}$$

Also, it follows from the above remarks about V that the central term in each subsequence has a maximum value at the equality (determined by h) chosen to form the subsequence. The question is where these subsequences can be located in V, and this is determined by the first symmetric subsequence of V (necessarily located at $v(2^t)$ for some integer $t > 1$) sufficiently long to contain the subsequence in question as in the above examples. First consider $\ell_h = 2m-1-2(h-1)$ for $1 \leq h \leq j$. Clearly

$$\begin{aligned} \ell_h &= 2^{q+1} - (2h-1) + 2j \\ &\geq 2^{q+1} - (2j-1) + 2j \\ &= 2^{q+1} + 1 \end{aligned}$$

and

$$\begin{aligned} \ell_h &\leq 2^{q+1} - 1 + 2j \\ &< 2^{q+1} - 1 + 2^{q+1} \quad = \quad 2^{q+2} - 1. \end{aligned}$$

Thus the first symmetric subsequence of V that contains the subsequences for these values of h is the one of length $2^{q+2}-1$ centered at $v(2^{q+2})$ since the longest shorter symmetric subsequence of V is only of length $2^{q+1}-1$. But this implies that, for these values of h,

$$r+2h-1 \equiv s-2h+3 \equiv 2^k \pmod{2^{k+1}}, \quad k > q+1$$

as claimed. Moreover, entirely similar arguments show that

$$2^{q+1}+1 \le \ell_h < 2^{q+2}-1 \ \text{ for } \ m+2-j \le h \le m+1$$

implying, as above, that

$$r+2h-1 \equiv s-2h+3 \equiv 2^k \pmod{2^{k+1}}, \quad k > q+1$$

for these values of h; and that

$$2^q+1 \le \ell_h \le 2^{q+1}-1 \ \text{ for } \ j < h \le \frac{m+1}{2}$$

and

$$2^q < \ell_h \le 2^{q+1}-1 \ \text{ for } \ \frac{m+1}{2} < h < m+2-j$$

which together imply that

$$r+2h-1 \equiv s-2h+3 \equiv 2^k \pmod{2^{k+1}}, \quad k \ge q+1$$

for $j < h < m+2-j$. Finally, if $j=0$, then $m = 2^q-1$ and we still have

$$\ell_h = \begin{cases} 2m+1-2(h-1), & 1 \le h \le \frac{m+1}{2} \\ 2h-1, & \frac{m+1}{2} < h \le m+1\,. \end{cases}$$

And again we can show, as above, that

$$2^q+1 \le \ell_h \le 2^{q+1}-1 \ \text{ for } \ 1 \le h \le \frac{m+1}{2}$$

and

$$2^q < \ell_h \le 2^{q+1}-1 \ \text{ for } \ \frac{m+1}{2} < h \le m+1$$

implying that

$$r+2h+1 \equiv s-2h+3 \equiv 2^k \pmod{2^{k+1}}, \quad k \ge q+1$$

for $1 \le h \le m+1$.

This completes the proof, and we note that similar results can be obtained for the hexagon considered here but rotated by 120° and 240°.

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