A RECURSIVE FORMULA FOR SUMS OF SQUARES

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1. INTRODUCTION

If t is a positive integer and n is a non-negative integer, let $r_t(n)$ denote the number of representations of n as a sum of t squares of integers. (Representations that differ only in order of terms are considered distinct.) A vast literature exists that is devoted to this subject. (See [3].)

In [1], Ewell used elementary means to obtain a formula for $r_3(n)$ in terms of $q_0(n)$, the number of self-conjugate partitions of n. Let the integer $t \geq 4$. In this note, we extend Ewell's result, obtaining a formula for $r_t(n)$ in terms of $r_{t-3}(n)$ and $q_0(n)$.

2. PRELIMINARIES

Let $t \geq 1$, $n \geq 0$.

$r_t(n)$ denotes the number of representations of n as a sum of t squares of integers.

$q(n)$ denotes the number of partitions of n into distinct parts (or into odd parts).

$q_0(n)$ denotes the number of partitions of n into distinct, odd parts (or the number of self-conjugate partitions of n).

$\omega(j) = \frac{j(3j-1)}{2}$ if $j \in Z$ (pentagonal numbers)

Identities

Let $x, z \in C$, $|x| < 1$, $z \neq 0$.

$$\prod_{n\geq 1}(1+x^{2n-1}) = \sum_{n\geq 0} q_0(n)x^n \tag{1}$$

$$\prod_{n\geq 1}(1+x^n) = \prod_{n\geq 1}(1-x^{2n-1})^{-1} = \sum_{n\geq 0} q(n)x^n \tag{2}$$

$$\prod_{n\geq 1}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2}z^n \tag{3}$$

$$\prod_{n\geq 1}\frac{(1-x^n)^3}{(1+x^n)^2} = \sum_{i=-\infty}^{\infty}(1-6i)x^{\omega(i)} \tag{4}$$

Remarks: Identities (1) and (2) are the well-known generating function identities for $q_0(n)$ and $q(n)$ respectively; (3) is the triple product identity; (4) is due to B. Gordon. (See [2] .)

3. THE MAIN RESULT

Theorem 1: If the integer $t \geq 4$, then

$$(-1)^n r_t(n) = \sum_{\omega(i)+j+k=n} (1-6i)(-1)^{j+k} r_{t-3}(j) q_0(k).$$

Remarks: In the sum defined above, we have $i \in Z$, j, $k \geq 0$.

Proof: Setting $z = -1$ in (3), we obtain

$$\prod_{n\geq 1}(1-x^{2n})(1-x^{2n-1})(1-x^{2n-1}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2}.$$

If we simplify, using (2), we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} = \prod_{n\geq 1} \frac{1-x^n}{1+x^n}. \tag{5}$$

If we raise equation (5) to the t power, we get

$$\sum_{n\geq 0}(-1)^n r_t(n) x^n = \prod_{n\geq 1} \left(\frac{1-x^n}{1+x^n}\right)^t. \tag{6}$$

Now

$$\prod_{n\geq 1} \left(\frac{1-x^n}{1+x^n}\right)^t = \prod_{n\geq 1} \frac{(1-x^n)^3}{(1+x^n)^2} \prod_{n\geq 1} \left(\frac{1-x^n}{1+x^n}\right)^{t-3} \prod_{n\geq 1}(1+x^n)^{-1}$$

$$= \prod_{n\geq 1} \frac{(1-x^n)^3}{(1+x^n)^2} \prod_{n\geq 1} \left(\frac{1-x^n}{1+x^n}\right)^{t-3} \prod_{n\geq 1}(1-x^{2n-1})$$

$$= \sum_{i=-\infty}^{\infty} (1-6i)x^{\omega(i)} \sum_{j\geq 0}(-1)^j r_{t-3}(j) x^j \sum_{k\geq 0}(-1)^k q_0(k),$$

invoking (4), (6), and (1). The conclusion now follows if we match coefficients of like powers of x. $\square$

Remarks: If we define $r_0(0) = 1$ and $r_0(n) = 0$ for $n \geq 1$, then Theorem 1 is also valid for $t = 3$, and reduces to Ewell's formula for $r_3(n)$.

REFERENCES

[1] J. A. Ewell. "Recursive Determination of the Enumerator for Sums of Three Squares." *Internat. J. Math. & Math. Sci.* **24** (2000): 529-532.

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[3] E. Grosswald. *Representations of Integers as Sums of Squares*, (1985), Springer-Verlag.

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